INTEGRAL DOMAINS IN WHICH NONZERO LOCALLY PRINCIPAL IDEALS ARE INVERTIBLE

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Abstract. We study locally principal ideals and integral domains, called LPI domains, in which every nonzero locally principal ideal is invertible. We show that a finite character intersection of LPI overrings is an LPI domain. Hence if a domain D is a finite character intersection $D = \cap D_P$ for some set of prime ideals of D, then D is an LPI domain.

Bazzoni in [10] and in [11] put forward the conjecture: If D is a Prüfer domain such that every nonzero locally principal ideal of D is invertible, then D is of finite character. (A domain D is *Prüfer* if every nonzero finitely generated ideal of D is invertible and D is of *finite character* if every nonzero nonunit of D belongs to only finitely many maximal ideals of D.) This conjecture was resolved in the affirmative by Holland, Martinez, McGovern, and Tesemma in [18]. Later Halter-Koch [16] stated and proved an analog of Bazzoni's conjecture for r-Prüfer monoids, which in the domain case are PVMD's (defined below) and include Prüfer domains. Recently, in [23], the second author has treated the Bazzoni Conjecture in a simpler manner encompassing the results of the above mentioned authors. This note is to record the results proved in an effort to answer the following question. What are the domains, called LPI domains, that have the property LPI: every nonzero locally principal ideal is invertible? Our main result is that a finite character intersection of LPI overrings is an LPI domain. Hence if D has a set S of prime ideals with $D = \cap_{P\in S} D_P$ being of finite character, D is an LPI domain. As our work will involve the use of star operations, we provide below a quick review.

Most of the information given below can be found in [22] and [13, sections 32, 34], also see [15]. Let D denote an integral domain with quotient field K and let $F(D)$ be the set of nonzero fractional ideals of D. A *star operation* $*$ *on* D is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$, and

(c) $(A^*)^* = A^*$.

For $A, B \in F(D)$ we define $*$-*multiplication* by $(AB)^* = (A^*B)^* = (A^*B^*)^*$. A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal A is of *finite type* if $A = B^*$ where B is a finitely generated fractional ideal. A star operation $*$ is said to be of *finite character* if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. For $A \in F(D)$ define $A^{-1} = \{x \in K \mid xA \subseteq D\}$ and call $A \in F(D)$ $*$-*invertible* if $(AA^{-1})^* = D$. Clearly every invertible ideal is $*$-invertible for every star operation $*$. If $*$ is of finite character and A is $*$-invertible, then A^* is of finite

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type. The best known examples of star operations are the *d-operation* defined by $A \mapsto A_d = A$, the *v-operation* defined by $A \mapsto A_v = (A^{-1})^{-1}$, and the *t-operation* defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$. Given two star operations $*_1, *_2$ on D we say that $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in F(D)$. Note that $*_1 \leq *_2$ if and only if $(A^{*_1})^{*_2} = A^{*_2}$ for all $A \in F(D)$, or equivalently, $(A^{*_2})^{*_1} = A^{*_2}$ for all $A \in F(D)$. By definition t is of finite character, $t \leq v$, and $\rho \leq t$ for every star operation ρ of finite character. If $*$ is a star operation of finite character, then using Zorn's Lemma we can show that a proper integral ideal maximal with respect to being a $*$-ideal is a prime ideal and that every proper integral $*$-ideal is contained in a maximal $*$-ideal. Let us denote the set of all maximal $*$-ideals by $*$-*max*(D). It can also be easily established that for a star operation $*$ of finite character on D, we have $D = \cap\{D_M \mid M \in *\text{-max}(D)\}$. A v-ideal A of finite type is t-invertible if and only if A is *t-locally principal*, i.e., for every $M \in t\text{-max}(D)$ we have AD_M principal. An integral domain D is called a *Prüfer v-multiplication domain (PVMD)* if every nonzero finitely generated ideal of D is t-invertible. According to Griffin [14, Theorem 5] D is a PVMD if and only if D_M is a valuation domain for each $M \in t\text{-max}(D)$. A domain D is said to be of *finite character* (resp., *finite t-character*) if every nonzero nonunit of D belongs to only a finite number of maximal ideals (resp., maximal t-ideals), or equivalently, the intersection $D = \cap\{D_M \mid M \in \max(D)\}$ (resp., $D = \cap\{D_M \mid M \in t\text{-max}(D)\}$) is of *finite character* or is *locally finite,* i.e., each nonzero element of D (or K) is a unit in almost all D_M. More generally, we say that D is of *finite prime character* (or *finite S-character*, if we need to mention the set S) if there exists a set S of prime ideals of D with $D = \cap_{P \in S} D_P$ locally finite. We can now state the PVMD analog of Bazzoni's conjecture which was proved by Halter-Koch [16] and the second author [23]: a PVMD D is of finite t-character if and only if every t-locally principal t-ideal of D is t-invertible.

Recall that an ideal I in a ring R is called a *cancellation ideal* if $IJ = IK$ for ideals J and K of R implies that $J = K$. Note that in this definition we can replace "$=$" by "$\subseteq$". In [7] it was shown that an ideal I is a cancellation ideal if and only if for each maximal ideal M of R, I_M is a regular principal ideal of R_M. With this in mind, we have the following characterization of nonzero locally principal ideals in an integral domain.

Theorem 1. *For an integral domain D and nonzero ideal I of D, the following conditions are equivalent.*

(1) *I is locally principal.*
(2) *I is a cancellation ideal.*
(3) *I is faithfully flat as a D-module.*

Proof. (1) $\Leftrightarrow$ (2) This is given in [7, Theorem] as mentioned in the previous paragraph. (1) $\Rightarrow$ (3) This follows since being faithfully flat is a local condition. (3) $\Rightarrow$ (2) Suppose that $IJ \subseteq IK$ for ideals J and K of D. Then $I \otimes_D ((J+K)/K) = I \otimes_D (J+K)/I \otimes_D K = I(J+K)/IK = IK/IK = 0$; so $(J+K)/K = 0$, that is, $J \subseteq K$. Here the first two equalities follow from the flatness of I and the last equality from the "faithfulness". □

With this preparation we return to the question: What integral domains satisfy the property LPI: every nonzero locally principal ideal is invertible? Now it is well known that a nonzero ideal is invertible if and only if it is finitely generated and locally principal [20, Theorems 58 and 62]. So this gives the criterion that a nonzero locally principal ideal is invertible if (and only if) it is finitely generated. Thus a Noetherian domain is an LPI domain. Also, according to Lemma 37.3 of Gilmer [13], stated below for integral domains (with our addition of the last statement), every semi-quasi-local domain is an LPI domain. In fact, in a semi-quasi-local domain a locally principal ideal is actually principal [20, Theorem 60]. However, the result that we would really like, namely that a locally principal ideal contained in only finitely many maximal ideals is invertible, is not true. For [13, Example 42.6] gives an example of an almost Dedekind domain D (i.e., D is locally a DVR) with exactly one noninvertible maximal ideal M. Thus M is contained in a unique maximal ideal and is locally principal, but M is not invertible.

Proposition 1. *([13, Lemma 37.3]) Let $x \in D$ such that x belongs to finitely many maximal ideals $M_1, M_2, ..., M_n$ of D. If A is an ideal of D containing x such that AD_{M_i} is finitely generated for each i between 1 and n, then A is finitely generated. (Hence if A is nonzero locally principal, A is invertible.)*

We next give a slight extension of Proposition 1 which has the added benefit that its converse is also true.

Theorem 2. *Let D be an integral domain and I a nonzero locally principal ideal (resp., locally finitely generated ideal) of D that is contained in only finitely many maximal ideals. Then I is invertible (resp. finitely generated) if and only if there exits a finitely generated ideal $A \subseteq I$ such that A is contained in only finitely many maximal ideals. In the invertible case I can be generated by two elements.*

Proof. ($\Rightarrow$) If I is finitely generated (which includes the invertible case), we can take A to be I. ($\Leftarrow$) Suppose that A is contained in the maximal ideals $M_1, M_2, ..., M_n$. If $I \not\subseteq M_i$ for some i, then for $x \in I - M_i$, we can replace A by (A, x) and $(A, x) \not\subseteq M_i$. Thus we can assume that $M_1, M_2, ..., M_n$ are also the maximal ideals containing I. For each i, choose a finite set $\{b_{ij}\}$ in I with $D_{M_i}(b_{ij}) = I_{M_i}$. Replace A by $(A, \{b_{1j}\}, \ldots, \{b_{nj}\})$. Then $A_{M_i} = I_{M_i}$ for each i and $A_N = D_N = I_N$ for each other maximal ideal N of D. So $A = I$ locally and hence globally. Thus I is finitely generated. Hence if I is locally principal, then I is finitely generated and locally principal and thus invertible. The last statement follows from [13, Exercise 9, Section 7]. □

What is really behind the fact that a nonzero finitely generated locally principal ideal is invertible is the fact that for ideals I and J of a commutative ring R with I finitely generated and any multiplicatively closed set S of R, we have $(J : I)_S = (J_S : I_S)$. We next give a characterization of invertible ideals using this. The last statement of the next theorem is a special case of [4, Theorem 12].

Theorem 3. (a) *For a nonzero locally principal ideal I of an integral domain D, the following conditions are equivalent.*

(1) *I is invertible.*
(2) *I is finitely generated.*
(3) *For all nonzero $x \in I$, $(Dx :_D I)_M = (D_M x :_{D_M} I_M)$ for each maximal ideal M of D.*

(4) *For some nonzero* $x \in I$, $(Dx :_D I)_M = (D_M x :_{D_M} I_M)$ *for each maximal ideal* M *of* D.
(5) $(I^{-1})_M = (I_M)^{-1}$ *for each maximal ideal* M *of* D.

(b) *Let* I *be a nonzero locally principal ideal of an integral domain* D. *Suppose that for some nonzero* $x \in I$, Dx *has a primary decomposition. Then* I *is invertible. Thus an integral domain in which every proper principal ideal has a primary decomposition is an LPI domain.*

Proof. (a) (1) ⇒ (2) ⇒ (3) ⇒ (4) Clear. (4) ⇒ (1) Let M be a maximal ideal of D. Now $((Dx : I)I)_M = (D_M x : I_M)I_M = D_M x$ where the last equality holds since I_M is principal. Thus $(Dx : I)I = Dx$ locally and hence globally. Since I is a factor of a nonzero principal ideal, I is invertible. (2) ⇒ (5) $(I^{-1})_M = [D :_K I]_M = [D_M :_K I_M] = (I_M)^{-1}$ where the second equality follows since I is finitely generated. (5) ⇒ (1) $(II^{-1})_M = I_M(I^{-1})_M = I_M(I_M)^{-1} = D_M$ for each maximal ideal M of D. Here the last equality holds because I_M is principal. So $II^{-1} = D$ and hence I is invertible. (b) Suppose that $Dx = Q_1 \cap \cdots \cap Q_n$ where each Q_i is primary. Then $(Dx : I) = (Q_1 \cap \cdots \cap Q_n : I) = (Q_1 : I) \cap \cdots \cap (Q_n : I)$. Now for any multiplicatively closed set S and any primary ideal Q we have $(Q : I)_S = (Q_S : I_S)$. (See, for example, [4, Lemma 11].) Hence $(Dx : I)_S = (Q_1 \cap \cdots \cap Q_n : I)_S = ((Q_1 : I) \cap \cdots \cap (Q_n : I))_S = (Q_1 : I)_S \cap \cdots \cap (Q_n : I)_S = (Q_{1S} : I_S) \cap \cdots \cap (Q_{nS} : I_S) = (Q_{1S} \cap \cdots \cap Q_{nS} : I_S) = (D_S x : I_S)$. By (a) I is invertible. □

While the condition that the nonzero locally principal ideal I contains a nonzero principal ideal having a primary decomposition is sufficient for I to be invertible, it is by no means necessary. For example, if V is a two-dimensional valuation domain and x is a nonzero element of V contained in a height-one prime ideal, then certainly Vx is invertible, but Vx contains no nonzero principal ideal having a primary decomposition.

Of course, using Proposition 1 or Theorem 2, we conclude that if D is of finite character, then nonzero locally principal ideals are invertible and can be generated by two elements. Note that the example of a Dedekind domain that is not a PID shows that we can not replace "two elements" by "one element" here or in Theorem 2.

Recall that if $\{D_\alpha\}$ is a family of overrings of D (rings between D and its quotient field K) such that $D = \cap D_\alpha$, then the function $A \mapsto \cap AD_\alpha$ is a star operation which is of finite character if $D = \cap D_\alpha$ is of finite character [2, (4) Theorem 2]. Also, observe that if I is a nonzero locally principal ideal of D and T is an overring of D (or more generally any integral domain containing D as a subring), then IT is (nonzero) locally principal. If $IT = T$, this is clear. Suppose that $IT \subseteq M$, a maximal ideal of T, and let $P = M \cap D$. Then ID_P is principal (being a localization of ID_N for any maximal ideal N of D containing P). Hence $(IT)_M = ID_P T_M$ is principal. Thus by Theorem 1, IT is also a cancellation ideal.

We have one more result to quote before we give the main result of this note.

Lemma 1. *(*[1, Theorem 2.1]*) Let* I *be a nonzero locally principal ideal in an integral domain. Then* I *is a* t*-ideal. Further,* I *is invertible if and only if* I *is of finite type.*

Note that while a nonzero locally principal ideal is a t-ideal, it need not be a v-ideal. (Of course, an invertible ideal is a v-ideal.) For example, if D is an almost Dedekind domain, then a maximal ideal M of D is locally principal, but M is a v-ideal if and only if M is invertible. Note that a locally principal v-ideal need not be invertible. For let D be an integral domain that is locally a GCD domain, but is not a GGCD domain. (Recall that D is a *GGCD domain* if the intersection of two invertible (or equivalently, principal) ideals is invertible.) So there are nonzero $x, y \in D$ with $(x) \cap (y)$ not invertible. But $(x) \cap (y)$ is a locally principal v-ideal, necessarily not of finite type. For an example of such a domain, see [17]. With this preparation we have our main result.

Theorem 4. *Let D be an integral domain where $D = \cap D_\alpha$ is a finite character intersection of overrings D_α each satisfying LPI. Then D is an LPI domain. Hence if D has finite prime character (i.e., D has a set S of prime ideals such that $D = \cap_{P \in S} D_P$ is of finite character), then D is an LPI domain.*

Proof. Let $*$ be the star operation given by $A \mapsto \cap AD_\alpha$. Since the intersection is of finite character, $*$ has finite character. Let I be a nonzero locally principal ideal of D. Choose $0 \neq x \in I$. Let $\alpha_1, \ldots, \alpha_n$ be the indices with $xD_\alpha \neq D_\alpha$. Now ID_{α_i} is nonzero locally principal and hence invertible and thus finitely generated. So we can enlarge Dx to a finitely generated ideal $A \subseteq I$ such that $ID_\alpha = AD_\alpha$ for each α. Thus $I^* = \cap ID_\alpha = \cap AD_\alpha = A^*$. Since $*$ has finite character and I is a t-ideal, we have $I = I_t = (I^*)_t = (A^*)_t = A_t$. So I has finite type. By Lemma 1, I is invertible. The second statement is now immediate because a quasi-local domain is an LPI domain. $\square$

Recall that an integral domain D is a *Mori domain* if it satisfies the ascending chain condition on divisorial ideals. So Mori domains include Noetherian domains and Krull domains. The reader may see [9] for an introduction to these rings and for a recent bibliography. For our purposes let us note that in a Mori domain D every maximal t-ideal is divisorial and from Theorem 3.3 of [9] we conclude that a Mori domain is of finite t-character.

Corollary 1. *In any integral domain of finite character or finite t-character, every nonzero locally principal ideal is invertible. Consequently, a Mori domain is an LPI domain.*

We next mention a result related to the second statement of Theorem 4. Note that this proposition can be used to give an alternate proof of the second statement of Theorem 4. For if we assume that the ideal A in Proposition 2 is nonzero locally principal, then A being t-invertible is of finite type. So by Lemma 1, A is invertible.

Proposition 2. *(*[6, Lemma 2.2]*) Let S be a collection of nonzero prime ideals of an integral domain D. Suppose that $D = \cap_{P \in S} D_P$ where the intersection has finite character. Let $*$ be the star operation $A \mapsto A^* = \cap_{P \in S} AD_P$. If $A \in F(D)$ such that AD_P is principal for each $P \in S$, then A is $*$-invertible and hence t-invertible.*

We next consider the question of when the polynomial ring $D[X]$ is an LPI domain. We show that if $D[X]$ is an LPI domain, then so is D and for D integrally closed, the converse is true. While we do not know in general whether D an LPI domain implies $D[X]$ is an LPI domain, we note that D has finite prime character (resp., finite t-character) if and only if $D[X]$ does.

Theorem 5. *Let D be an integral domain.*

(1) *If $D[X]$ is an LPI domain, then so is D. If D is an integrally closed LPI domain, then $D[X]$ is an LPI domain.*
(2) *D has finite prime character if and only if $D[X]$ has finite prime character.*
(3) *D has finite t-character if and only if $D[X]$ has finite t-character.*

Proof. (1) Suppose that $D[X]$ is an LPI domain. Let A be a nonzero locally principal ideal of D. Then by the remarks of the paragraph preceding Lemma 1, $AD[X]$ is a nonzero locally principal ideal of $D[X]$. Since $D[X]$ is an LPI domain, $AD[X]$ is invertible. Hence A is invertible. So D is an LPI domain. Conversely, suppose that D is an integrally closed LPI domain. Let B be a nonzero locally principal ideal of $D[X]$. Since D is integrally closed and B is a t-ideal, $B = \frac{f}{g}AD[X]$ where $f, g \in D[X]$ and A is a t-ideal of D (see, for example, [3, Corollary 3.1]). Now B nonzero locally principal implies $AD[X]$ is nonzero locally principal and hence A is nonzero locally principal. (For let M be a maximal ideal of D. Then $AD[X]_{(M,X)}$ is principal and thus $AD[X]_{M[X]} = AD_M(X)$ is principal. But then AD_M is principal.) Since D is an LPI domain, A is invertible. Thus $AD[X]$ is invertible and hence so is B. So $D[X]$ is an LPI domain.

(2) ($\Leftarrow$) Suppose that $D[X]$ has finite S-character: $D[X] = \cap_{Q\in S} D[X]_Q$, locally finite. Let $S' = \{Q \cap D | Q \in S\}$. Then $D = \cap_{P\in S'} D_P$ is a finite S'-character representation for D. For certainly this representation is locally finite. And $D = \cap_{P\in S'} D_P$ since $D = D[X] \cap K = (\cap_{Q\in S} D[X]_Q) \cap K = \cap_{Q\in S}(D[X]_Q \cap K) \supseteq \cap_{P\in S'} D_P \supseteq D$ since $D[X]_Q \cap K \supseteq D_P$ where $P = Q \cap D$. ($\Rightarrow$) Suppose that D has finite S-character, so $D = \cap_{P\in S} D_P$ is locally finite. Now $D[X] = \cap_{P\in S} D_P[X]$ (but is not locally finite) and $D_P[X] = D[X]_{P[X]} \cap K[X]$. So $D[X] = (\cap_{P\in S} D[X]_{P[X]}) \cap (\cap_{Q\in T} D[X]_Q)$ is a finite prime character representation where $T = \{Q \in Spec(D[X]) | Q \cap D = 0\}$. The locally finiteness follows since for $\frac{f}{g} \in K(X)$, the contents $c(f)$ and $c(g)$ of f and g, respectively, are contained in only finitely many P; so $\frac{f}{g}$ is a unit in almost all $D[X]_{P[X]}$.

(3) This is given for D integrally closed in [21, Proposition 4.2]; but the hypothesis that D is integrally closed is not used. We offer a simpler proof. Recall from [19, Proposition 1.1] that if M is a maximal t-ideal of $D[X]$ with $M \cap D \neq 0$, then $M = (M \cap D)[X]$. Noting that if A is an integral t-ideal of D, then $A[X]$ is an integral t-ideal of $D[X]$, we conclude that A is a maximal t-ideal of D if and only if $A[X]$ is a maximal t-ideal of $D[X]$. Now suppose that $D[X]$ is of finite t-character and take a nonzero nonunit $d \in D$. Then d belongs to only a finite number of maximal t-ideals of $D[X]$, each necessarily of the form $M[X]$ where M is a maximal t-ideal of D. So D is of finite t-character. Conversely, suppose that D is of finite t-character. Let f be a nonzero nonunit of $D[X]$. Then f belongs to two kinds of maximal t-ideals M: (a) M such that $M \cap D \neq 0$ and (b) M such that $M \cap D = 0$. Now the ones in (a) are finite in number because D is of finite t-character and the ones in (b) being uppers to zero are also finite in number. $\square$

We opened this paper by noting that a Prüfer domain D is an LPI domain if and only if D has finite character (or equivalently, finite t-character, since every nonzero ideal of a Prüfer domain is a t-ideal). Thus examples of non-LPI domains include the ring of all algebraic integers and the ring of entire functions. Also, an almost Dedekind domain is an LPI domain if and only if it is Dedekind.

Let D be an integral domain such that for each $n \geq 1$ every proper principal ideal of $D[X_1, \ldots, X_n]$ has a primary decomposition (e.g., D is Noetherian). Then for any set of indeterminates $\{X_\alpha\}$, every proper principal ideal of $D[\{X_\alpha\}]$ has a primary decomposition and hence $D[\{X_\alpha\}]$ is an LPI domain. In particular, $\mathbb{Z}[\{X_\alpha\}]$ is an LPI domain. However, every ring is a homomorphic image of $\mathbb{Z}[\{X_\alpha\}]$ for some set of indeterminates $\{X_\alpha\}$. Thus the homomorphic image of an LPI domain need not be an LPI domain.

Let D be an integral domain and let $X^{(1)}(D)$ be the set of height-one prime ideals of D. Then D is said to be *weakly Krull* if $D = \cap_{P \in X^{(1)}(D)} D_P$ where the intersection is locally finite. Note that D is weakly Krull if and only if every nonzero proper principal ideal of D has a reduced primary decomposition involving only height-one primes [5, Theorem 13]. Thus for a one-dimensional integral domain D the following are equivalent: (1) D has finite character, (2) D has finite t-character, (3) D is weakly Krull, (4) every proper principal ideal of D has a primary decomposition, and (5) D is *Laskerian*, i.e., every proper ideal of D has a primary decomposition. Note that D can be of finite prime character without each proper principal ideal having a primary decomposition. For example, a valuation domain V is of finite character and finite t-character, but each proper principal ideal of V has a primary decomposition if and only if $dimV = 1$. However, no example comes to mind of a domain D in which every proper principal ideal has a primary decomposition, but D is not of finite prime character. Also, we have no example of an LPI domain that is not of finite prime character.

We next give an example of a ring of finite character (and hence an LPI domain) that is not of finite t-character. Thus an LPI domain need not be of finite t-character. Note that conversely, if D has finite t-character, then D need not have finite character as seen by $D = K[X_1, \ldots, X_n]$ where K is a field and $n \geq 2$. Thus $K[X_1]$ has finite character, but $K[X_1][X_2]$ does not; so in Theorem 5 we can not add (4) D is of finite character if and only if $D[X]$ is.

Example 1. *Let R be a regular local ring of dimension greater than 1 with quotient field K and let X be an indeterminate over K. Then $D = R + XK[X]$ is a domain that is of finite character and hence every locally principal nonzero ideal is invertible, but not of finite t-character. In fact, D contains a nonzero ideal A that is t-locally principal, but not t-invertible.*

Illustration: By Theorem 4.21 of [12] every maximal ideal of D is either of the form $M + XK[X]$ where M is the maximal ideal of R or is of the form $f(X)D$ where $f(0) = 1$. Now a typical element of $D = R + XK[X]$ is of the form $\frac{a}{b}X^r(1 + Xf(X))$ where $b \mid a$ if $r = 0$. If $r = 0$ and $c = \frac{a}{b} \in R$, then in $c(1 + Xf(X))$, c is clearly comaximal with $1 + Xf(X)$, $c \in M + XK[X]$, and $1 + Xf(X)$ is a product of principal (maximal) primes. We conclude that $c(1 + Xf(X))$ belongs to only a finite number of maximal ideals. Next if $r > 0$ we note that as X does not belong to any prime ideal of the form $(1 + Xg(X))D$, $\frac{a}{b}X^r$ does not belong to any maximal ideals containing $1 + Xf(X)$, which means that $1 + Xf(X)$ and $\frac{a}{b}X^r$ are comaximal. Noting that $\frac{a}{b}X^r \in M + XK[X]$, which is unique in this case, we conclude that $\frac{a}{b}X^r(1 + Xf(X))$ belongs to only a finite number of maximal ideals. Having exhausted all the cases we conclude that D is of finite character and consequently every nonzero locally principal ideal of D is invertible. To see that D is not of finite t-character recall from page 437 of [12] for P a nonzero prime of R, $P+XK[X]$ is a maximal t-ideal of D if and only if P is a maximal t-ideal of R. As R is a UFD with infinitely many non-associate principal primes we have infinitely many distinct maximal t-ideals of the form $pR+XK[X]$ containing the element X. To construct the ideal A select a set $\{p_1, p_2, ...\}$ of non-associate principal primes of R and as in [23] we can construct $A = (\{p_1^{-1} \cdots p_n^{-1}X\}_{n=1}^{\infty})$ which is t-locally principal, yet not of finite type and hence not t-invertible.

We end by considering the question: If D is an LPI domain and S is a multiplicative set of D, must D_S be an LPI domain? Note that if every proper principal ideal of D has a primary decomposition, then the same is true of D_S. Note, however, that if D has finite t-character, then D_S need not again be of finite t-character. In [8, Example 2c] is an example of a domain D of finite t-character with a maximal ideal M such that D_M does not have finite t-character. But as in the above mentioned example the result is still a quasi-local ring we do not have a definite counterexample to the question. However, if D is of finite t-character and t-Spec(D) is treed then the answer is yes for every multiplicative set S. Thus we have a simple special case. Moreover, we do not know of an example of an integral domain of finite prime character such that some localization D_S does not have finite prime character and we know of no example of an LPI domain with a localization that is not an LPI domain Thus we end with the following questions.

Question 1. If D is an LPI domain, is D of finite prime character?

Question 2. If D is an LPI domain, is $D[X]$ an LPI domain?

Question 3. If D is an LPI domain and S is a multiplicatively closed subset of D, is D_S an LPI domain?

Question 4. If every proper principal ideal of D has a primary decomposition, is D of finite prime character?

Question 5. If D has finite prime character and S is a multiplicatively closed subset of D, does D_S have finite prime character?

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