BASES OF PRE-RIESZ GROUPS AND CONRAD'S F-CONDITION

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Dedicated to the memory of Paul Conrad.

Abstract. Let $\mathcal{L}(S)$ denote the set of lower bounds of a set S in partially ordered set T, and let G^+ denote the positive cone of a partially ordered group G. We study directed groups G with the (pR) property: if $x_1, x_2, ..., x_n \in G^+$ such that $\mathcal{L}(x_1, x_2, ..., x_n) \neq \mathcal{L}(0)$ then there is a strictly positive element $l \leq x_i$ in G. Calling these groups pre-Riesz, we show that Conrad's F-condition which was stated for lattice ordered groups can still be stated for pre-Riesz groups and has similar effects modulo minor changes in definitions of basic elements and bases. As applications of our work we study integral domains whose groups of divisibility and groups of $*$-invertible $*$-ideals, for finite character star operations $*$, are pre-Riesz and pre-Riesz satisfying Conrad's F-condition.

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1. Introduction

We plan to study the pre-Riesz groups and Conrad's F-condition with the backdrop of partially ordered (p.o. for short) groups associated with an integral domain D, as that is where the study originated. First of these is the, well known, group of divisibility $G(D)$. This works well with our plan, as we plan to study the other p.o. groups associated with D in light of our results on $G(D)$. As the definitions of others will require introduction we postpone the mention and concentrate on the group of divisibility $G(D)$ of an integral domain D. We shall look into the situations when $G(D)$ is a lattice ordered group, a Riesz group (a directed p.o. group with Riesz interpolation property) or a pre-Riesz group (a directed p.o. group such that every finite set of strictly positive elements with at least one non-negative lower bound has a strictly positive lower bound) and in each case we shall study the effects of the so-called F-condition of Paul Conrad's. A lattice ordered group G is said to satisfy Conrad's F-condition if every strictly positive element of G exceeds at most a finite set of (mutually) disjoint elements. We plan to introduce the relevant ring theoretic terminology along the way and show the relevance of our study of the group of divisibility to other p.o. groups that arise from various notions of invertibility of ideals.

In section 2 we put together group and order theoretic preliminaries, with an emphasis on $G(D)$. In section 3 we introduce pre-Riesz groups as a generalization of Riesz groups and hence of lattice ordered groups. In this section we look at Conrad's work on basic elements, bases and Conrad's F-condition in the framework of lattice ordered groups and generalize these notions to pre-Riesz groups, showing finally that a pre-Riesz group satisfying Conrad's F-condition has a basis. In section 4, we introduce the notion of a star operation and study integral domains with pre-Riesz group of divisibility. It turns out that these domains are precisely the domains known as the PSP domains. We also show, assuming familiarity with star operations here, that a PSP domain D is of finite t-character if and only if $G(D)$ satisfies Conrad's F-condition. This treatment is a bit elaborate but we plan to use it to set the stage for section 5. Assuming familiarity with ideal systems, for now, we plan to study in section 5, the groups $Inv_*(D)$ of $*$-invertible $*$-ideals of D under $*$-multiplication where $*$ is a star operation of finite type. We show that $Inv_*(D)$ is a directed p.o. group and study the domains for which $Inv_*(D)$ is pre-Riesz and show that in this case $Inv_*(D)$ satisfies Conrad's F-condition if and only if D is of finite $*$-character. We also characterize domains D such that every maximal $*$-ideal of D contains a $*$-invertible $*$-ideal that belongs to no other maximal $*$-ideal.

2. Preliminaries

Let D be an integral domain with quotient field K. For $x, y \in K\backslash\{0\}$, we say that x divides y with respect to D if there exists $a \in D$ such that $y = ax$. Usually we use $x \mid y$ or if a reference to D is important, we use $x \mid_D y$ to denote the fact that x divides y with respect to D. Now $y = ax$ means $yD = axD \subseteq xD$. Thus $x \mid_D y$ if and only if $yD \subseteq xD$. Clearly since for all $x, y, z \in K^\times = K\backslash\{0\}$, $x \mid_D x$ and $x \mid_D y$ and $y \mid_D z$ implies $x \mid_D z$ we conclude that $\mid_D$ is a preorder on K. However we note that $x \mid_D y$ and $y \mid_D x$ implies that $xD = yD$. Now $xD = yD$ if and only if x and y are associates of each other. Thus $x \mid_D y$ can be made into a partial order on $G(D) = K^{\times}/U(D)$, where $U(D)$ denotes the set of units of D, by defining $xU(D) \leq yU(D)$ if and only if $xD \supseteq yD$. We shall use U for $U(D)$.

It is now easy to see that if for any $x, y, z \in G(D)$ and $x \leq y$ we readily have $zx \leq zy$, (because $xD \supseteq yD$ implies $zxD \supseteq zyD$). That is, the partial order is compatible with multiplication in $G(D)$. We already know that $G(D) = K^{\times}/U$ is a group under multiplication $xU * yU = xyU$. Now recall that a group G is called a p.o. group if G is partially ordered such that the partial order is compatible with the group operation: if $x \leq y$ in G we have $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $z \in G$, where $\cdot$ is the binary operation of G. We shall mainly be concerned with Abelian groups. Because of the compatibility condition we have $x \leq y$ if and only if $y^{-1} \leq x^{-1}$, in a p.o. group.

Notes on p.o. groups ([10] will be our main reference for p.o. groups):

(1). We note that U is the identity of $G(D)$, and the positive cone of $G(D)$, $G(D)^+ = \{xU \in G(D) : xD \subseteq D\} = \{xU : x \in D\backslash\{0\}\}$. As, for each $x \in K\backslash\{0\}$, xU can be identified with xD, one can also define the group of divisibility as $G(D) = \{kD : k \in K\backslash\{0\}\}$. A D-submodule A of K is called an integral ideal if $A \subseteq D$. So $G(D)^+$ can be identified with the set of (nonzero) integral principal ideals of D.

(2). The group of divisibility $G(D)$ of an integral domain D is a somewhat specialized p.o. group. Specialized in that $G(D)$ is upper directed, i.e., for all $hU, kU \in G(D)$ there is at least one $lU \in G(D)$ such that $hU, kU \leq lU$. We can define "lower directed" in a dual fashion. For a p.o. group being upper directed is the same as being lower directed, hence every group of divisibility $G(D)$ is a directed group. However, note that there are directed groups which are not groups of divisibility of any domain [21, pp. 394-395].

(3). Usually $+$ is preferred as the notation for the binary operation in a p.o. group, even when the binary operation is non-commutative. Respecting that we shall use $+$ as the operation in $G(D)$, and often replace the identity U by zero 0. But for $hU, kU \in G(D)$, $hU + kU = hkU$. If you adopt the $G(D) = \{kD : k \in K^{\times}\}$, care must be taken to avoid regarding $hD + kD$ as the sum of two ideals, which may not make much sense in $G(D)$, unless D is a B,zout domain (every two or finitely generated ideal is principal). Note that B,zout domains are GCD domains, but the converse is not true, for instance, $Z[x]$ is a GCD domain which is not a B,zout domain since $gcd(x, 2) = 1$ in $Z[x]$, but x and 2 do not generate $Z[x]$; here Z is the ring of integers.

(4). If $S = [A, \leq]$ is a partially ordered set (poset) then $S' = [A, \leq']$, such that $a \leq' b$ if and only if $b \leq a$, is also a poset called the dual of S. To make a dual of a statement involving only $\leq$ or $\geq$ all we have to do is change $\leq$ into $\geq$ and vice versa, and make appropriate changes in terminology.

(5). In a poset P it is important to know if a subset S has upper bounds, elements that exceed every element of S. In notation $\mathcal{U}(S) = \{x \in P : x \geq s, \forall s \in S\}$, which is called the set of upper bounds of S. If $\mathcal{U}(S) = \Phi$ we say that S is not bounded from above. We define the set of lower bounds $\mathcal{L}(S)$ dually and the corresponding definitions. So $\mathcal{L}(S) = \{x \in P : x \leq s, \forall s \in S\}$ is the set of lower bounds of S, $\mathcal{L}(S) = \Phi$ denotes S is not bounded from below.

(6). A set $S \subseteq P$ is said to have a least upper bound (lub) or supremum (sup) if there is $b \in \mathcal{U}(S)$ such that $b \leq x$ for all $x \in \mathcal{U}(S)$. That is, $\mathcal{U}(S)$ has a least element. It is easy to see that if $lub(S) = b$ exists, then it is unique. It is easy to see that $lub(S) = b$ if and only if

$\mathcal{U}(S) = \mathcal{U}(\{b\})$; we usually denote $\mathcal{U}(\{b\})$ by $\mathcal{U}(b)$. If S consist of only two elements x, y then $lub(x, y)$, if it exists in P, is called the join of x and y and is denoted by $x \vee y$. The symbol $\vee$ is called "join". We define greatest lower bound (glb) or infimum (inf) dually and note that $glb(S) = c$ if and only if $\mathcal{L}(S) = \mathcal{L}(c)$. We denote $glb(x, y)$, if it exists in P, by $x \wedge y$ and call $\wedge$ the meet symbol.

Formulas for upper/lower bounds of a subset S of a p.o. group G:

(1) For all $a \in G$ we have $a + \mathcal{U}(S) = \mathcal{U}(a + S) = \mathcal{U}(\{a + s : s \in S\})$. Thus if x, y, $x \vee y \in G$, then for all $a \in G$ we have $a + (x \vee y) = (a + x) \vee (a + y)$.

(2) For all $a \in G$ we have $a + \mathcal{L}(S) = \mathcal{L}(a + S) = \mathcal{L}(\{a + s : s \in S\})$. Thus if x, y, $x \wedge y \in G$, then for all $a \in G$ we have $a + (x \wedge y) = (a + x) \wedge (a + y)$.

(3) If S is a subset of a p.o. group G and $-S = \{-x : x \in S\}$, then $-\mathcal{U}(S) = \mathcal{L}(-S)$. Thus if $x, y, x \vee y \in G$ then $-(x \vee y) = -x \wedge -y$.

(4) $-\mathcal{L}(S) = \mathcal{U}(-S)$. Thus if $x, y, x \vee y \in G$ then $-(x \wedge y) = -x \vee -y$.

(5) $\mathcal{L}(a, b) = a - \mathcal{U}(a, b) + b = a + b - \mathcal{U}(a, b)$ for all $a, b \in G$. So $a \wedge b = a + b - (a \vee b)$, if $a \vee b \in G$.

(6) $\mathcal{U}(a, b) = a - \mathcal{L}(a, b) + b = a + b - \mathcal{L}(a, b)$ for all $a, b \in G$. So $a \vee b = a + b - (a \wedge b)$, if $a \wedge b \in G$.

Notes on lattice ordered groups:

Of interest are the sets P such that every pair of elements x, y in P has an lub in P i.e. for all $x, y \in P$, $x \vee y$ exists and is in P. Such a set P is called a $\vee$-semilattice or a join semilattice. Similarly we define a meet semilattice ($\wedge$-semilattice). $P = \{\{a, b\}, \{a, c\}, \{a, b, c\}\}$ with inclusion as the partial order can serve as a join semilattice that is not a meet semilattice, and $Q = \{\Phi, \{a\}, \{b\}\}$ is a meet semilattice that is not a join semilattice. For a p.o. group, however, being a join semilattice is equivalent to being a meet semilattice and vice versa, because of formulas (5) and (6). A poset P that is a meet as well as a join semilattice is called a lattice. So a p.o. group that is a meet or a join semilattice is a lattice and hence is called a lattice ordered (l.o.) group. In an l.o. group the meet and join distribute over each other.

The simplest examples of l.o. groups come from totally ordered Abelian groups. Recall that a poset P is totally ordered if for all $x, y \in P$ we have $x \leq y$ or $y \leq x$, so $x = x \vee y$ or $y = x \vee y$. We say that two elements x, y of a poset P are comparable if $x \leq y$ or $y \leq x$, and denote by $x \parallel y$ if they are not comparable. So a poset P is totally or linearly ordered if every two elements of P are comparable. Now a valuation domain V with quotient field K is defined by the property that for all $x, y \in K \backslash \{0\}$, $x \mid y$ ($yV \subseteq xV$) or $y \mid x$ ($xV \subseteq yV$). So $G(V)$ is totally ordered and it is easy to see that if $G(D)$ is totally ordered then D is a valuation domain. On the other hand, W. Krull showed that for a given totally ordered Abelian group G, there exists a valuation domain V such that the group of divisibility $G(V)$ is isomorphic to G.

Next $G(D)$ is lattice ordered if and only if D is a GCD domain (for every pair $x, y \in D \backslash \{0\}$, $xD \cap yD = mD$ is principal). If D is a GCD domain it is easy to see that $mU = xU \vee yU$ and $\text{GCD}(x, y) = \frac{xy}{m} (= xU + yU - xU \vee yU) = x \wedge y$. Now to show that a directed group G is lattice ordered it is sufficient to show that G^+ is a lattice [10, Proposition 3, p. 13] . Conversely, if $G(D)$ is lattice ordered then for each pair $xU, yU \in G(D)^+$ we have $mU = xU \vee yU$ and it is easy to see that $mD = xD \cap yD$, for each pair $x, y \in D\backslash\{0\}$, so D is a GCD domain. On the other hand, P. Jaffard and J. Ohm generalized Krull's result to show that for a given l.o. Abelian group G, there exists a B,zout domain D such that the group of divisibility $G(D)$ is isomorphic to G. Note that in the context of monoids and (generalized) divisor theories, Geroldinger and Halter-Koch got a nice proof of the Jaffard-Ohm correspondence in [12]. Furthermore, Rump and Yang [17] gave a categorical interpretation of the Jaffard-Ohm correspondence and established a general extension theorem for valuations with values in an abelian l-group, which yields a proof of Anderson's conjectural refinement of the Jaffard-Ohm theorem.

3. The bases and Conrad F-condition in pre-Riesz groups

We call two positive elements x, y in a p.o. group G disjoint if for every $z \leq x, y$ we have $z \leq 0$, the identity of G, i.e. $x \wedge y = 0$. It is well known that in a p.o. group G if $a \leq b + c$ and $a \wedge b = 0$ with $c \geq 0$, then $a \leq c$. (Adding c to both sides of $a \wedge b = 0$ we have $(a + c) \wedge (b + c) = c$. Now $a \leq a + c, b + c$ implies $a \leq a + c \wedge b + c = c$.). Also if a, b are disjoint then $a + b = a \vee b$.

If x, y are two elements of a p.o. group G and if $d = x \wedge y$ then $x - d$ and $y - d$ are disjoint (subtract d from both sides.)

An l.o. group G has the following property: For $x, a_1, a_2 \in G^+$, if $x \leq a_1 + a_2$ then for some $b_1, b_2 \in G^+$, $x = b_1 + b_2$ where $b_1 \leq a_1$ and $b_2 \leq a_2$. (For this let $b_1 = x \wedge a_1$. Then $x - b_1 \leq (a_1 - b_1) + a_2$ which gives $x - b_1 \leq a_2$, because $(x - b_1) \wedge (a_1 - b_1) = 0$. Setting $x - b_1 = b_2$ we have the result.)

For a general p.o. group G call an element $x \in G^+$ primal if for all $a_1, a_2 \in G^+$**,** $x \leq a_1 + a_2$ implies that $x = b_1 + b_2$, for some $b_1, b_2 \in G^+$ such that $b_i \leq a_i$. This term appeared in the p.o. group context in [25]. So, if G is an l.o. group every positive element of G is primal. A directed p.o. G is called a Riesz group if every element of G^+ is primal. For examples of Riesz groups that are not l.o. groups we refer to [11, 19, 24]. The most prominent characterizing property of Riesz groups is the (m, n)-interpolation property also called the Riesz interpolation property (RIP): given $x_1, x_2, ..., x_m; y_1, y_2, ..., y_n \in G$ with $x_i \leq y_j$ for all integers $i \in [1, m], j \in [1, n]$ there is $z \in G$ such that

$$x_i \leq z \leq y_j \tag{3.1}$$

for all integer pairs $(i, j) \in [1, m] \times [1, n]$.

Using the RIP we can prove the following result.

Proposition 3.1. *In a Riesz group G the following property holds. (pR): If $0 < x_1, x_2, ..., x_n \in G$ with $\mathcal{L}(x_1, x_2, ..., x_n) \neq \mathcal{L}(0)$, then there exists $r \in G$ such that $0 < r \leq x_1, x_2, ..., x_n$.*

Proof. If $\mathcal{L}(x_1, x_2, ..., x_n) \neq \mathcal{L}(0)$ then there is at least one $g \in \mathcal{L}(x_1, x_2, ..., x_n)$ such that $g \nleq 0$. Thus we have $0, g \leq x_1, x_2, ..., x_n$ and by RIP there is r such that $0, g \leq r \leq x_1, x_2, ..., x_n$. Now $r \geq 0$ and $r \neq 0$ because of g. □

An integrally closed integral domain whose group of divisibility is a Riesz group was introduced as a Schreier domain in [6] and its more general form, an integral domain whose group of divisibility is Riesz was discussed in [23] as a pre-Schreier domain. Translating directly from the definition of a Riesz group, D is a pre-Schreier domain if and only if for every triplet $x, y, z \in D\backslash\{0\}$, $x \mid yz$ implies $x = x_1x_2$ where $x_1 \mid y$ and $x_2 \mid z$.

Call a directed p.o. group G a pre-Riesz group if G satisfies the property (pR). Conrad's F-condition [7] on l.o. groups reads: Each strictly positive element x in an l.o. group G is greater than at most a finite number of (mutually) disjoint elements.

Let us, briefly, see how Conrad proceeded in [7] and what he achieved.

For G an l.o. group call $x \in G$ basic if $x > 0$ and $[0, x]$ is a chain i.e. all h, k with $0 \leq h, k \leq x$ are comparable. Call a set $S \subseteq G$ disjoint if for every pair of distinct strictly positive elements $x, y \in S$, $x \wedge y = 0$. A set that is maximal w.r.t. being disjoint and which contains basic elements only is called, in [7], a basis of G. Not every l.o. group has a basis and we have included a simple example (Example 3.6) towards the end of this section.

Conrad in [7, Lemma 4.1] showed that a nonempty subset S of an l.o. group G is a basis if and only if S is disjoint and $(S\backslash\{s\}) \cup \{x, y\}$ is non-disjoint for any $s \in S$ and for any $\{x, y\} \subseteq (G\backslash S) \cup \{s\}$, with $x \neq y$.

He calls a subset S of an l.o. group G independent if S is disjoint and if every element of S is basic and he indicates, via Zorn's Lemma, that a nonempty independent set is contained in a maximal independent set [7, Lemma 5.2]. He also shows [7, Theorem 5.1] that an l.o. group G has a basis if and only if every strictly positive element in G exceeds at least one basic element. Moreover every basis of G is a maximal independent set and every maximal independent set is a basis provided that G has a basis. Finally he shows [7, Theorem 5.2] that if an l.o. group G satisfies condition F then G has a basis.

In a recent paper [16] Mott, Rashid and the second author redid, using their experience with factorization in pre-Schreier domains, Conrad's work, as far as the effect of Conrad's F-condition is concerned, for Riesz groups. We now investigate the effects of the F-condition in Pre-Riesz groups. In what follows we use $\mathcal{G}$ to denote a pre-Riesz group for the sake of distinction.

First let us note that for $x, y \in \mathcal{G}^+$, $x \wedge y \neq 0$ means that there exists $0 < r \leq x, y$. Here $x \wedge y \neq 0$ does not presume that $x \wedge y$ exists.

Call $x \in G^+$ a homogeneous element if for all $h, k \in [0, x]$, $h \wedge k = 0$ implies that $h = 0$ or $k = 0$. Since the only property of Riesz groups used in [16] was the property $(*)$ we conclude that the results on homogeneous elements proved in [16] hold for pre-Riesz groups. Call two homogeneous elements x, y of G related if $x \wedge y \neq 0$.

We list below relevant results from [16, Proposition 2.1].

Proposition 3.2. *Let $\mathcal{G}$ be a pre-Riesz group and let $x, y \in \mathcal{G}^+$. Then the following hold.*

(1) $x > 0$ is a homogeneous element if and only if $(0, x]$ is lower directed, i.e., for all $a, b \in (0, x]$ there is $t \in (0, x]$ such that $t \leq a, b$. So if x is homogeneous and there is h with $0 < h \leq x$ then h is homogeneous. Thus $x > 0$ is not homogeneous if and only if there is at least one pair $c, d \in (0, x]$ such that $c \wedge d = 0$.

(2) If $x \wedge y = 0$ and there is a $u \in \mathcal{G}$ with $0 < u \leq y$ then $x \wedge u = 0$. Consequently if $x, y > 0$ and $x \wedge y = 0$, then for each pair $(i, j) \in (0, x] \times (0, y]$ we have $i \wedge j = 0$.

(3) Suppose that h and k are two homogeneous elements of $\mathcal{G}$. Then the following are equivalent:

(a) $h \wedge k = 0$, *(b) for each pair* $(a,b) \in (0,h] \times (0,k]$, $a \wedge b = 0$, *(c) there is at least one pair* $(a,b) \in (0,h] \times (0,k]$ *for which* $a \wedge b = 0$.

(4) Suppose that h *and* k *are two homogeneous elements of* $\mathcal{G}$*. Then the following are equivalent:*

(r) $h \wedge k \neq 0$, *i.e.,* h *and* k *are related (s) for each pair* $(a,b) \in (0,h] \times (0,k]$ *we have* $a \wedge b \neq 0$, *(t) there is at least one pair* $(a,b) \in (0,h] \times (0,k]$ *such that* $a \wedge b \neq 0$.

(5) Relatedness is an equivalence relation on the set of all homogeneous elements.

(6) If $x, y \in \mathcal{G}^+$ *and* $x \wedge y = 0$ *and if* h *is a homogeneous element then* h *must be disjoint with at least one of* x, y*. More generally if there are mutually disjoint positive elements* $b_1, b_2, ..., b_n$ *then* h *must be disjoint with at least* $n-1$ *of the* b_i.

Now call a subset S of $\mathcal{G}$ a basis if S is a maximal disjoint set consisting of homogeneous elements.

Lemma 3.3. *A nonempty subset* S *of a pre-Riesz group* $\mathcal{G}$ *is a basis if and only if* S *is disjoint and* $(S \backslash \{s\}) \cup \{x,y\}$ *is non-disjoint for any* $s \in S$ *and for any* $\{x,y\} \subseteq (\mathcal{G} \backslash S) \cup \{s\}$, *with* $x \neq y$.

Proof. Let S be a basis and suppose that for some $s \in S$, $(S \backslash \{s\}) \cup \{x,y\}$ is disjoint for some $\{x,y\} \subseteq (G \backslash S) \cup \{s\}$, with $x \neq y$. But since S is maximal disjoint $x \wedge s \neq 0$, $y \wedge s \neq 0$. This leads to the existence of $0 < t \leq x, s$ and $0 < u \leq y, s$ and to the existence of $0 < w \leq t, u, x, y$ which contradicts the assumption that $x \wedge y = 0$. Conversely suppose that S is disjoint and satisfies the conditions set in the lemma. If $S \cup \{x\}$ is disjoint for some $x \in G \backslash S$, then $S \backslash \{s\} \cup \{s, x\}$ is non-disjoint and $s \neq x$, a contradiction. If $s \in S$ and s is not homogeneous then there exists at least one pair of elements $0 < x, y < s$ such that $x \wedge y = 0$. But then $x, y \notin S$ and $x \neq y$ and $(S \backslash \{s\}) \cup \{x,y\}$ is disjoint, a contradiction. Thus S is a maximal disjoint set consisting of homogeneous elements. ☐

Call a set $S \subseteq \mathcal{G}$ independent if S consists of mutually disjoint homogeneous elements and note that if we have an ascending chain $\{F_\alpha\}_{\alpha \in I}$ of independent subsets of $\mathcal{G}$ under inclusion then the union of such a chain is again an independent set. For if $x \in \cup F_\alpha$ then x must be homogeneous because x belongs to one of F_α. Also if there are two elements $x, y \in \cup F_\alpha$ with $x \wedge y \neq 0$ then for some $\alpha \in I$ we have $x, y \in F_\alpha$. This observation leads to the following statement.

Lemma 3.4. *Let* S *be a nonempty independent set in a pre-Riesz group* $\mathcal{G}$*. Then there is a maximal independent set containing* S.

Theorem 3.5. *A nontrivial pre-Riesz group* $\mathcal{G}$ *has a basis if and only if (P): each* $0 < x \in G$ *exceeds at least one homogeneous element. Every basis of* $\mathcal{G}$ *is a maximal independent set and every maximal independent subset of* $\mathcal{G}$ *is a basis provided* $\mathcal{G}$ *has a basis.*

Proof. Let $S = \{0 < a_\alpha : \alpha \in I\}$ be a basis for G and consider $0 < x \in G$. There must exist $\alpha \in I$ such that $x \wedge a_\alpha \neq 0$, for otherwise $x \wedge a_\alpha = 0$ for all $\alpha \in I$ and that will contradict the maximality of S as a disjoint set. Now $x \wedge a_\alpha \neq 0$ implies that there is $0 < h \leq x, a_\alpha$. Because a_α is homogeneous, h is homogeneous. Hence x exceeds at least one homogeneous element. That S is a maximal independent subset of G is obvious. Conversely, suppose that G satisfies the property (P). By Lemma 3.4, there is a maximal independent subset $T = \{0 < a_\alpha : \alpha \in I\}$ of G

and by the property $T \neq \phi$. All we need show is that T is a maximal disjoint set. Suppose on the contrary that there is an element $0 < x \in G$ such that $x \wedge a_\alpha = 0$ for all $\alpha \in I$. But then by property (P), x exceeds a homogeneous element h and by (2) of Proposition 3.2, h is disjoint with a_α for all $\alpha \in I$, contradicting the choice of T as a maximal independent subset of G. □

Now we give the promised example of a lattice ordered and hence a pre-Riesz group without basis. (This example is adapted from http://www.lohar.com/mithelpdesk/hd 0307.pdf)

Example 3.6. *Let $S = \{X^\alpha : \alpha \in {}^+\}$ where ${}^+$ denotes the set of nonnegative rational numbers and let K be an algebraically closed field with 0 characteristic. Also let R be the semi-group ring $K[S] = \{\sum_{i=1}^{n} c_i X^{\alpha_i} : c_i \in K$ and $\alpha_i \in {}^+\}$. Then the group of divisibility $G(R)$ of R is a l.o. group without a basis.*

Illustration: Note that R can be regarded as an ascending union of the PIDs $R_{n!} = K[X^{\frac{1}{n!}}]$ where $n!$ denotes the factorial of the natural number n. That is $R = \bigcup R_{n!}$, where obviously $R_{n!} \subseteq R_{(n+1)!}$ for all natural numbers n. Being an ascending union of PIDs, R is a Bezout domain and so the group of divisibility of R is a lattice ordered group. To show that $G(R)$ does not have a basis all we need is a strictly positive element a of $G(R)$ such that a exceeds no basic element. Or equivalently every strictly positive element b below a exceeds at least two disjoint strictly positive elements. Translating, we need a nonzero nonunit a in R such that every nonzero nonunit factor b of a has at least two coprime nonunit factors. We claim that $X - 1$ is the required element. To establish this we show that $(X-1)$ is a product of nonassociate primes in $R_{n!}$ for each n. Having done that we would have shown that every nonunit factor of $(X-1)$ is a product of nonassociate primes in $R_{n!}$ for some n. For this we note that in $R_{n!} = K[X^{\frac{1}{n!}}]$, $X - 1 = (X^{\frac{1}{n!}})^{n!} - 1$.

Now the following general lemma will help.

Lemma 3.7. *Let K be a field with characteristic 0 and let X be an indeterminate over K and let $D = K[X]$. Then $(X^n - 1)D$ is a radical ideal for every natural number n.*

Proof. We first show that if $(f(X))^m$ divides $(X^n - 1)$ then $m = 1$. To see this suppose that $(X^n - 1) = (f(X))^m g(X)$. (Then clearly, $f(0) \neq 0 \neq g(0)$. Differentiating both sides, with respect to X, we get $nX^{n-1} = m(f(X))^{m-1} f'(X) g(X) + (f(X))^m g'(X)$ which forces $(f(X))^{m-1}$ to divide nX^{n-1}. But this is possible only if $(f(X))^{m-1}$ is a unit, which means that $m = 1$. From this it follows that $(X^n - 1)$ is a product of distinct (mutually non associated) primes of $K[X]$. (We have adopted this proof for its simplicity and direcness. Otherwise, as one of the referees points out, the proof is complete if we say that: In characteristic 0, cyclotomic polynomials have no multiple zeros.) □

Note here that as K is algebraically closed all those nonassociate primes are linear polynomials in $R_{n!}$. Now suppose that h is a nonunit factor of $(X-1)$ in R and that there is a nonunit $f \in R$ such that for some m we have $f^m \mid h$. This indeed means that $f^m \mid (X-1)$. Since R is an ascending union of $\{R_{n!}\}$, f is a polynomial in $R_{k!}$ for some k and so $f^m \mid ((X^{\frac{1}{k!}})^{k!} - 1)$ and by Lemma 3.7, $m = 1$. Thus d itself is a product of nonassociate primes in $R_{k!}$ for some suitable k and so is divisible by at least two coprime elements of R.

Theorem 3.8. *If $\mathcal{G}$ satisfies Conrad's condition F then $\mathcal{G}$ has a basis.*

Proof. Suppose that the condition holds but G has no basis. Then by Theorem 3.5 there is at least one $0 < y \in G$ such that y exceeds no homogeneous element. This y is clearly non-homogeneous. Thus there are at least two disjoint elements p_1, q_1 with $y > p_1, q_1 > 0$. None of p_1, q_1 exceeds a homogeneous element for otherwise y would. So, say, $p_1 > p_2, q_2 > 0$ so that $p_2 \wedge q_2 = 0$. Since $p_1 \wedge q_1 = 0$ and $p_1 > q_2$ we have $q_1 \wedge q_2 = 0$. Next $p_2 > p_3, q_3 > 0$ such that $p_3 \wedge q_3 = 0$. Again since $p_i's$ are disjoint with $q_i's$ we conclude that q_1, q_2, q_3 are mutually disjoint. Similarly producing $q's$ using $p's$ and using induction we can produce an infinite sequence $\{q_i\}$ of mutually disjoint elements less than y. Contradicting the assumption that G satisfies F. □ □

Corollary 3.9. *For a pre-Riesz group $\mathcal{G}$ the following are equivalent:*

(1) $\mathcal{G}$ satisfies Conrad's F condition

(2) every strictly positive element exceeds at least one and at most a finite number of homogeneous elements.

Proof. (1) $\Rightarrow$ (2) is direct. For (2) $\Rightarrow$ (1), suppose that (2) holds yet G does not satisfy (1). Then there is $0 < x \in G$ that exceeds an infinite sequence $\{x_i\}$ of mutually disjoint strictly positive elements of G. Now each of x_i exceeds at least one homogeneous element h_i. Because $\{x_i\}$ are mutually disjoint so are $\{h_i\}$ a contradiction. □

4. Integral domains D with $G(D)$ pre-Riesz

To study integral domains with pre-Riesz groups of divisibility, we need to review the so-called star operations on D. For this let $F(D)$ denote the set of nonzero fractional ideals of D, i.e. D submodules A of K such that there is $d \in D\backslash\{0\}$ such that $dA \subseteq D$. In terms of the group of divisibility the fractional ideals A of D are among the subsets of $G(D)$ that are bounded from below. (For $c, d \in D\backslash\{0\}$, $\frac{c}{d}D \leq A \Leftrightarrow A \subseteq \frac{c}{d}D \Rightarrow dA \subseteq cD \subseteq D$. Also to each nonempty subset S of $G(D)$ you can assign a D submodule $SD = \{\sum_{i=1}^{n} s_i d_i : s_i \in S$ and $d_i \in D\}$ of K generated by S and indeed SD is a fractional ideal if and only if S is bounded from below.)

Most of the information given below can be found in [26, 13]. A star operation $*$ on D is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(a_*) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b_*) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(c_*) $(A^*)^* = A^*$.

Remark 4.1. *Note that conditions (b_*) and (c_*) are essentially the axioms of a closure operator on a join semilattice, hence they can be characterized by a single equation:*

$$A \cup (A^* \cup B^*)^* = (A \cup B)^* \tag{4.1}$$

For a proof of (4.1) in detail the reader is referred to [20].

For $A, B \in F(D)$ we define $*$-multiplication by $(AB)^* = (A^*B)^* = (A^*B^*)^*$ and $*$-addition by $(A+B)^* = (A^*+B)^* = (A^*+B^*)^*$. A fractional ideal $A \in F(D)$ is called a $*$-ideal if $A = A^*$ and a $*$-ideal of finite type if $A = B^*$ where B is a finitely generated fractional ideal. Clearly a principal fractional ideal is a $*$-ideal for every star operation $*$ by (a_*). Also note that if $\{A_\alpha\}$

is a family of $*$-ideals, for an operation $*$, such that $\cap A_\alpha \neq (0)$ then $B = \cap A_\alpha$ is a $*$-ideal. ($B = \cap A_\alpha \Rightarrow B \subseteq A_\alpha$ for each $\alpha \Rightarrow B^* \subseteq A_\alpha$ for each $\alpha \Rightarrow B^* \subseteq \cap A_\alpha = B$. Because $B \subseteq B^*$ already we have $B = B^*$.)

A star operation $*$ is said to be of finite character if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$ for all $A \in F(D)$. To ensure that, for $*$ of finite character, $0 \neq A$ is a star ideal it is enough to check that for each nonzero finitely generated ideal $I \subseteq A$ we have $I^* \subseteq A$. To each star operation $*$ we can associate an operation $*_s$ defined by $A^{*_s} = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$, for all $A \in F(D)$. It is easy to see that for a finitely generated $A \in F(D)$ we have $A^{*_s} = A^*$. Using this fact we can verify that $*$ is of finite character if and only if $*_s = *$. So, for this reason, we shall use $*_s$ as a prototype for a star operation of finite character.

For $A \in F(D)$ define $A^{-1} = \{x \in K \mid xA \subseteq D\} = \{x : xa \in D$ for each $a \in A\backslash\{0\}\} = \{x : x \in \bigcap_{a \in A\backslash\{0\}} a^{-1}D\} = \bigcap_{a \in A\backslash\{0\}} a^{-1}D$. Clearly $A \in F(D)$ implies $A^{-1} \in F(D)$. The most well known examples of star operations are: the v-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the t-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$, and the d-operation defined by $d : A \mapsto A$. The v-operation can also be equivalently defined as $A \mapsto A_v = \bigcap_{(x \in K^\times) \wedge (A \subseteq xD)} xD$, [10, page 101]. Now note that by (a$_*$) and (b$_*$) $A \subseteq xD$ implies that for any star operation $A^* \subseteq xD$. Thus by the previous comment $A^* \subseteq A_v$ for every star operation $*$. Next A_v, being an intersection of principal fractional ideals, is a $*$-ideal, i.e. $(A_v)^* = A_v$ for every star operation $*$. Also note that if $A \subseteq B$ then $A^{-1} \supseteq B^{-1}$ so $A^{-1} \supseteq (A^*)^{-1}$. So $A_v \subseteq (A^*)_v$, but $A^* \subseteq A_v$ implies that $(A^*)_v \subseteq (A_v)_v = A_v$. Thus $(A^*)_v = A_v$ which gives $(A^*)^{-1} = A^{-1}$, and so $(A_v)^{-1} = A^{-1}$. But as $(A_v)^{-1} = ((A^{-1})^{-1})^{-1}$, we have $(A^{-1})_v = A^{-1}$.

Given two star operations $*_1, *_2$ we say that $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in F(D)$. Note that $*_1 \leq *_2$ if and only if $(A^{*_1})^{*_2} = (A^{*_2})^{*_1} = A^{*_2}$. By definition t is of finite character, $t \leq v$ while $\rho \leq t$ for every star operation ρ of finite character. If $*$ is a star operation of finite character then using Zorn's Lemma we can show that a proper integral ideal maximal w.r.t. being a star ideal is a prime ideal and that every proper integral $*$-ideal is contained in a maximal $*$-ideal. Let us denote the set of all maximal $*$-ideals by $* - max(D)$. It can also be easily established that for a star operation $*$ of finite character on D we have $D = \bigcap_{M \in * - \max(D)} D_M$. Another star operation that is gaining popularity these days is the so called w-operation defined by $A \mapsto A_w = \bigcap_{M \in t - \max(D)} AD_M$. This star operation received detailed treatment by McCasland and Wang in [18]. In this paper the w-operation was equivalently defined as $A_w = \{x \in K : xJ \subseteq A$ for some finitely generated ideal J with $J^{-1} = D\}$. The w-operation is also of finite character.

Now we are looking at integral domains D such that $G(D)$ is a pre-Riesz group. Thus for $x_1U, x_2U, ..., x_nU \in G(D)^+$ with $\mathcal{L}(x_1U, x_2U, ..., x_nU) \neq \mathcal{L}(U)$ there must be $x \in D\backslash\{0\}$ such that $D < xU \leq x_1U, x_2U, ..., x_nU$. This means that if

$$\mathcal{L}(x_1U, x_2U, ..., x_nU) \neq \mathcal{L}(U)$$

then there is a nonunit $x \in D\backslash\{0\}$ such that $xD \supseteq x_1D, x_2D, ..., x_nD$, i.e.

$$(x_1, x_2, ..., x_n) \subseteq xD.$$

So we need to find ring-theoretic meanings of $\mathcal{L}(x_1U, x_2U, ..., x_nU) \neq \mathcal{L}(U)$. But this inequality means that there is $tU = \frac{u}{v}U$ for $\frac{u}{v} \in K\backslash\{0\}$ such that $\frac{u}{v}U \leq x_iU$ but $\frac{u}{v}U \nleq U$. That is $\frac{u}{v}D \supseteq x_iD$ but $\frac{u}{v}D \nsupseteq D$, i.e., $(x_1, x_2, ..., x_n) \subseteq \frac{u}{v}D$ where $u \nmid v$. This forces $(x_1, x_2, ..., x_n)_v \neq D$, because if $(x_1, x_2, ..., x_n)_v = D$ then as $(x_1, x_2, ..., x_n) \subseteq \frac{u}{v}D$, this would force $D = (x_1, x_2, ..., x_n)_v \subseteq \frac{u}{v}D$ resulting in $u \mid v$ a contradiction. On the other hand if $(x_1, x_2, ..., x_n)_v \neq D$ then there exists $\frac{u}{v} \in K\backslash\{0\}$ with $u \nmid v$ such that $(x_1, x_2, ..., x_n) \subseteq \frac{u}{v}D \nsupseteq D$. (For if there is none such then via $\bigcap_{\substack{y \in K\backslash\{0\} \\ (x_1,...,x_n) \subseteq yD}} yD = (x_1, x_2, ..., x_n)_v$ we would be forced to admit that $(x_1, x_2, ..., x_n)_v = D$.) This can of course be translated back to $\mathcal{L}(x_1U, x_2U, ..., x_nU) \neq \mathcal{L}(U)$. Thus $\mathcal{L}(x_1U, x_2U, ..., x_nU) \neq \mathcal{L}(U)$ if and only if $(x_1, x_2, ..., x_n)_v \neq D$. Consequently we can make the following statement.

Proposition 4.2. *An integral domain D has pre-Riesz group of divisibility if and only if for all $x_1, x_2, ..., x_n \in D\backslash\{0\}$ with $(x_1, x_2, ..., x_n)_v \neq D$ there is a non-unit x of D such that $(x_1, x_2, ..., x_n) \subseteq xD$.*

Now recall that a set $x_1, x_2, ..., x_n \in D\backslash\{0\}$ is said to be coprime (v-coprime) if $(x_1, x_2, ..., x_n) \subseteq xD$ implies that x is a unit (respectively if $(x_1, x_2, ..., x_n)_v = D$). From the above description it is clear that a domain D has pre-Riesz group of divisibility if and only if for every finite set $x_1, x_2, ..., x_n \in D\backslash\{0\}$, $x_1, x_2, ..., x_n$ coprime implies that $x_1, x_2, ..., x_n$ are v-coprime. Recall also that a polynomial $f = \sum_{i=0}^{n} f_iX^i \in D[X]$ is called primitive (super primitive) if its coefficients are coprime (v-coprime). A domain D over which Primitive polynomials are Super Primitive is called a PSP domain. These domains have been around for some time, see for instance [2, 3]. It is easy to see that in a PSP domain coprime implies v-coprime.

In the case of PSP domains the F-condition can be translated to: Every nonzero nonunit is divisible by at most a finite number of mutually coprime elements. Indeed a homogeneous element in a PSP domain can be defined as a nonzero nonunit x such that every pair r, s of nonunit factors of x is non (v-) coprime. So Corollary 3.9 can be restated as the following result.

Corollary 4.3. *Let D be a PSP domain. Then $G(D)$ satisfies Conrad's F-condition if and only if every nonzero nonunit of D is divisible by at least one and at most a finite number of mutually coprime homogeneous elements.*

In a PSP domain, Conrad's F-condition can be put to another use; as was done in [22] for GCD domains.

Remark 4.4. *Let D be a PSP domain. Note that if x is any element of $D\backslash\{0\}$ such that x is coprime with a homogeneous element t similar to r then x must be coprime to r. This is because t similar to r means there is a nonunit $s \mid t, r$. So if x is coprime to t, x is coprime to s. Suppose on the contrary that x is non-coprime to r then there is a homogeneous element u such that $u \mid x, r$. Since x is coprime to s, u is coprime to s, resulting in two coprime nonunits dividing r contradicting that r is homogeneous.*

Lemma 4.5. *If $x_1, x_2, ..., x_n, r$ are nonzero nonunits of a PSP domain, with r a homogeneous element such that each of x_i is non-coprime with r then there is a homogeneous element s, similar to r such that $(x_1, x_2, ..., x_n) \subseteq sD$.*

Proof. Because x_1, r are not coprime there is a nonunit r_1 such that $(x_1, r) \subseteq r_1D$. Using the fact that if x_2 is coprime to r_1 then x_2 must be coprime to r (by Remark 4.4) we conclude that there must be a nonunit r_2 in D such that $(x_2, r_1) \subseteq r_2D$, repeating the above argument over and over we get nonunits $r_3, ..., r_n$ where each r_i divides the preceding one and thus we have $(x_1, x_2, ..., x_n) \subseteq r_nD$ where r_n is a nonunit factor of r and hence a homogeneous element similar to r. □

Lemma 4.6. *Let D be a PSP domain and let r be a homogeneous element of D. Then the set $P(r) = \{x \in D$: such that x is non coprime with $r\}$ is a maximal t-ideal of D.*

Proof. Clearly $P(r)$ contains all homogeneous elements similar to r and the 0 element. By Lemma 4.5, $P(r)$ is an ideal such that for all $x_1, x_2, ..., x_n \in P(r)\backslash\{0\}$ there is a homogeneous element s similar to r such that $(x_1, x_2, ..., x_n) \subseteq sD$ and hence $(x_1, x_2, ..., x_n)_v \subseteq sD \subseteq P(r)$ from which it follows that $P(r)$ is a t-ideal. To see that $P(r)$ is a maximal t-ideal let Q be an integral ideal properly containing $P(r)$. But then $x \in Q\backslash P(r)$ is coprime with r ensuring that Q is not a t-ideal. (Since in a PSP domain, coprime is v-coprime.) □

Let us call the prime ideal $P(r)$ of Lemma 4.6 the prime ideal associated with the homogeneous element r. Call an integral domain of finite character (finite t- character) if every nonzero nonunit of D belongs to at most a finite number of maximal ideals (maximal t-ideals).

Proposition 4.7. *Let D be a PSP domain such that $G(D)$ satisfies Conrad's F-condition. Then (a) every maximal t-ideal of D is associated to a homogeneous element and (b) D is of finite t-character.*

Proof. (a) Let Q be a maximal t-ideal of D and let $x \in Q\backslash\{0\}$. Then because of the F-condition there are finitely many mutually coprime homogeneous elements $r_1, r_2, ..., r_n$ dividing x, by Corollary 4.3. We can assume that n is the largest such number. If Q is associated to one of the r_i, we have nothing to prove. Suppose now that Q does not contain any homogeneous element. Then as $Q \neq P(r_i)$, there is $y \in Q\backslash \cup P(r_i)$. So y is coprime to each of r_i. On the other hand $(x, y)_v \neq D$ and so x and y must have a common factor t. This t has a homogeneous factor h by Lemma 3.3 and because y is coprime to all of r_i, t is coprime to all of r_i contradicting the assumption that $r_1, r_2, ..., r_n$ are all the mutually coprime homogeneous elements. Since this contradiction has arisen from the assumption that Q does not contain a homogeneous element we have the conclusion. The part (b) is now obvious. □

Instead of saying "$G(D)$ satisfies Conrad's F-condition" we may just say that D satisfies Conrad's F-condition.

Because $G(D)$ is an l.o. group $\Rightarrow$ $G(D)$ is a Riesz group $\Rightarrow$ $G(D)$ is a pre-Riesz group we conclude that D a GCD domain $\Rightarrow$ D a pre-Schreier domain $\Rightarrow$ D is a PSP domain. So, Proposition 4.7 can be restated for pre-Schreier domains and for GCD domains.

Remark 4.8. *It is well known and well documented that there are Riesz groups that are not l.o. groups, see e.g.* [11], *and that there are pre-Schreier domains that are not GCD domains, see e.g.* [24]. *Indeed there are examples (see* [2]*) of PSP domains that are not pre-Schreier.*

Below we give two examples of PSP domains, one that is of finite t-character and one that is not.

Example 4.9. *Let Z be the ring of integers, let R be the field of real numbers, and let X be an indeterminate over R. Then $D = Z + XR[[X]]$ is a PSP domain that is not a pre-Schreier domain and does not satisfy Conrad's F-condition.*

Example 4.10. *$D = Z_{pZ} + XR[[X]]$ is a PSP with only one maximal ideal and D obviously satisfies Conrad's F-condition.*

Illustration. Example 4.9: Note that every maximal ideal of $D = Z + XR[[X]]$ is principal of the form $pZ + XR[[X]] = pD$ and so a t-ideal. Now if $x_1, x_2, ..., x_n \in D$ such that $(x_1, x_2, ..., x_n)_v \neq D$ then $(x_1, x_2, ..., x_n)_v$ must be contained in one of the pD. So, D is a PSP domain. That D is not of finite (t-) character follows from the fact that X is divisible by all the primes in Z. To see that D is not pre-Schreier note that $\pi X \mid X^2 = X \cdot X$ but πX cannot be written as $\pi X = x_1 x_2$ such that $x_i \mid X$. Example 4.10 obviously does not need any illustration.

Next we tackle the question: if D is a PSP domain of finite t-character must D satisfy Conrad's F-condition?

Lemma 4.11. *Let D be a PSP domain. Then $x \in D$ is a homogeneous element if and only if x belongs to a unique maximal t-ideal.*

Proof. If x belongs to a unique maximal t-ideal P then all nonunit factors of x belong to P and hence are non-coprime. So, x is a homogeneous element. Conversely, suppose that x is a homogeneous element and that x belongs to two maximal t-ideals P, Q. Let $y \in P \backslash Q$ and $z \in Q \backslash P$. Then as P, Q are maximal t-ideals $(y, Q)_t = D = (z, P)_t$. So there are $q_1, q_2, ..., q_r \in Q$ such that $(y, q_1, q_2, ..., q_r)_v = D$. Now as q_i share Q with x, $(x, q_1, q_2, ..., q_r)_v \neq D$, hence there is a nonunit t such that $(x, q_1, q_2, ..., q_r) \subseteq tD$. Being a factor of a homogeneous element t is homogeneous. Because t is a factor of q_i, t must be coprime with y. On the other hand y shares P with x and hence there is a nonunit common factor u of x and y, which leads to a homogeneous element x having two coprime factors t and u a contradiction. □

Proposition 4.12. *If D is a PSP domain of finite t-character then D satisfies Conrad's F-condition.*

Proof. If D does not satisfy Conrad's F-condition then there is an element $x \in D \backslash \{0\}$ such that x is divisible by infinitely many mutually coprime elements. Now since a maximal t-ideal cannot contain a pair of v-coprime elements there have to be infinitely many maximal t-ideals containing x, a contradiction. □

Alternate proof. Let $x \in D \backslash \{0\}$ and let $P, P_1, P_2, ..., P_n$ be the set of all the (distinct) maximal t-ideals containing x. Then there are $y_i \in P \backslash P_j$ such that each of y_i belongs to P and to no other maximal t-ideal. Now $(x, y_1, ..., y_n) \subseteq P$ and to no other maximal ideal. But since P is a t-ideal $(x, y_1, ..., y_n)_v \neq D$. But then by the PSP property there must be a y such that $(x, y_1, ..., y_n) \subseteq yD$. But then y belongs to P and to no other maximal t-ideal. So by Lemma 4.11, y is homogeneous and because P is a maximal t-ideal we conclude that $P = P(y)$. So each of the maximal t-ideals of D is associated to a homogeneous element. Now each nonzero nonunit of D is divisible by at least one and by the finite t-character property at most a finite number of mutually coprime homogeneous elements. We have the result by Corollary 4.3. □

We have concentrated more on the PSP domains so that they can serve as a prototype for more general considerations, when we consider other p.o. groups related to integral domains.

5. Groups arising from notions of invertibility of ideals

The other p.o. groups that we have in mind come from the various notions of invertibility of an ideal. We shall as above study those partially ordered groups and see what happens when the group under consideration is an l.o. group, a Riesz group or a pre-Riesz group and how a domain responds to any of these groups satisfying Conrad's F-condition.

For a star operation $*$ call $A \in F(D)$ $*$-invertible if there is a $B \in F(D)$ such that $(AB)^* = D$.

Remark 5.1. *$(AB)^* = D$ implies $(AA^{-1})^* = D$ implies $B^* = A^{-1}$; $A^* = A_v$. Since $* \leq v$ for each $*$ operation, A being $*$-invertible implies A being v-invertible. So a $*$-invertible $*$-ideal is a v-invertible v-ideal.*

Illustration. Indeed B is obviously contained in A^{-1}. So $AB \subseteq AA^{-1} \subseteq D$. Applying $*$ to both sides we have $D = (AB)^* \subseteq (AA^{-1})^* \subseteq D$. Next, multiplying $(AB)^* = D$ on both sides by A^{-1} and applying $*$ we get $(A^{-1})^* = (A^{-1}(AB)^*)^* = (A^{-1}AB)^* = ((A^{-1}A)^*B)^* = B^*$, since A^{-1} is a v-ideal we conclude that $B^* = A^{-1}$. From this it also follows that if $A \in F(D)$ is $*$-invertible then $A^* = A_v$. This is because in $(AA^{-1})^* = D$, $A^* = (A^{-1})^{-1} = A_v$. For the last part note that $(AA^{-1})^* = D$ implies $((AA^{-1})^*)_v = D$ which implies $(AA^{-1})_v = D$.

Clearly every invertible ideal is $*$-invertible for every star operation $*$.

Remark 5.2. *If $*$ is of finite character and A is $*$-invertible, then there is a finitely generated ideal $I \subset A$ such that $I^* = A^*$ and a finitely generated $J \subseteq A^{-1}$ such that $J^* = A^{-1}$.*

Illustration. This can be established as follows: If $*$ is of finite character then $(AB)^* = D$ implies that there is a finite set $\{x_1, x_2, ..., x_n\} \subseteq AB$ such that

$$(x_1, x_2, ..., x_n)^* = D.$$

Now as each of $x_i = \sum_{j=1}^{r_i} a_{ij} b_{ij}$ where $a_{ij} \in A$ and $b_{ij} \in B$. Thus there exist $\alpha_1, \alpha_2, ..., \alpha_r \in A$ and $b_1, ..., b_r \in B$ such that $x_i \in (\alpha_1, \alpha_2, ..., \alpha_r)(b_1, ..., b_r)$ for all i. So $(x_1, x_2, ..., x_n) \subseteq ((\alpha_1, \alpha_2, ..., \alpha_r)(b_1, ..., b_r)) \subseteq (\alpha_1, \alpha_2, ..., \alpha_r)B \subseteq AB$.

Since $(x_1, x_2, ..., x_n)^* = (AB)^* = D$ we have $((\alpha_1, \alpha_2, ..., \alpha_r)B)^* = D$. Multiplying both sides by A and applying $*$ we get $(\alpha_1, \alpha_2, ..., \alpha_r)^* = A^*$. Similarly there is $J = (\beta_1, \beta_2, ..., \beta_s) \subseteq B$ such that $J^* = B^* = A^{-1}$.

Remark 5.3. *Because for every star operation $*$ of finite character $* \leq t$ we conclude that if A is $*$-invertible for a finite type $*$ then A is t-invertible. This is because $(AA^{-1})^* = D$ implies $D = ((AA^{-1})^*)_t = (AA^{-1})_t$.*

Let $Inv_*(D) = \{A \in F(D) : A \text{ is a } *\text{-invertible } *\text{-ideal}\}$.

Proposition 5.4. *For an integral domain D, and a star operation $*$ defined on D, the set $Inv_*(D)$ is an Abelian directed p.o. group under $*$-multiplication ordered by $A \leq B \Leftrightarrow A \supseteq B$.*

The following lemma will be of help in proving the above proposition and in later considerations.

Lemma 5.5. *Let $A \in F(D)$ be a $*$-invertible ideal. Then the following hold.*

(1). For any $B, C \in F(D)$, $AB \subseteq AC$ implies that $B^* \subseteq C^*$.

(2). If $B \in F(D)$ is such that $B \subseteq A^*$ then there is an integral ideal C such that $B^* = (AC)^*$. Conversely if there is such an integral ideal C with $B^* = (AC)^*$ then $B \subseteq A^*$.

(3). If $B \in F(D)$ is $*$-invertible, then so is AB with inverse $(A^{-1}B^{-1})^*$.

(4). If $B \in F(D)$ is $*$-invertible and for some $C \in F(D)$, $B^* = (AC)^*$, then C is $*$-invertible.

(5). If $B \in F(D)$ is $*$-invertible, then there exists a $*$-invertible ideal $C \subseteq AB$ such that $C \subseteq A^*, B^*$.

(6). If $B \in F(D)$ is $*$-invertible and $A + B$ is $*$-invertible, then so is $A^* \cap B^*$.

Proof. (1). Multiplying both sides of $AB \subseteq AC$ by A^{-1} we get $A^{-1}AB \subseteq A^{-1}AC$. Now apply the $*$-operation to get $B^* \subseteq C^*$.

(2) Let $B \subseteq A^*$. Multiply by A^{-1} on both sides to get $A^{-1}B \subseteq A^{-1}A^* \subseteq D$. Set $C = A^{-1}B$. Multiplying both sides of $C = A^{-1}B$ by A and applying $*$ we get $B^* = (AC)^*$. Conversely because C is integral $AC \subseteq A$. So $B^* = (AC)^* \subseteq A^*$.

(3). Multiply AB by $A^{-1}B^{-1}$ and apply the $*$-operation.

(4). Multiply both sides of $B^* = (AC)^*$ by A^{-1} and apply $*$ to get

$$(A^{-1}B)^* = C^*.$$

By (3) $A^{-1}B$ is $*$-invertible. (This result holds more generally in monoids. We thank one of the referees for reminding us of that.)

(5). Any $d \in (A \cap B)\backslash\{0\}$ would prove the statement and, as one of the referees pointed out, A and B do not have to be $*$-invertible for this to hold. Yet we keep (5) as it is to keep the flow of the paper smooth.

(6) Note that, because A, B are $*$-invertible, $A^* = A_v, B^* = B_v$. Now consider $(A^{-1}B^{-1}(A+B))^* = (A^{-1}AB^{-1} + A^{-1}BB^{-1})^* = (B^{-1} + A^{-1})^*$, (because A and B are $*$-invertible). Next because $A, B, A+B$ are $*$-invertible $(B^{-1}+A^{-1})^*$ must be $*$-invertible. Now $((B^{-1}+A^{-1})^*)^{-1} = (B^{-1} + A^{-1})^{-1} = (A_v \cap B_v) = (A^* \cap B^*)$. Thus $(A^{-1}B^{-1}(A+B)(A^* \cap B^*))^* = D$. □

Proof of Proposition 5.4. That $Inv_*(D)$ is closed under $*$-multiplication follows from (3) of Lemma 5.5. The associativity and commutativity are inherited from the associativity of multiplication of fractional ideals. Next D is the identity and the existence of inverse is a part of the definition. So, $Inv_*(D)$ is an Abelian group. Now $Inv_*(D)$ is partially ordered by inclusion and so by its dual $\leq$. To see that $\leq$ is compatible with $*$-multiplication we let $A, B, X \in Inv_*(D)$ and let $A \leq B$. Then $A \supseteq B$ and by the usual properties of fractional ideals of D we have $AX \supseteq BX$ and so $(AX)^* \supseteq (BX)^*$ and so $(AX)^* \leq (BX)^*$. That $Inv_*(D)$ is directed follows from (5) of Lemma 5.5. □

Note that when $* = d$, $Inv_d(D)$ is the group of invertible ideals of D, when $* = v$, $Inv_v(D)$ is the group of v-invertible v-ideals of D. Next when $* = t$, $Inv_t(D)$ is the group of t-invertible

t-ideals of D. In short with the order $\leq$ defined as above we have each of these groups partially ordered and directed.

Now we look into what makes $Inv_*(D)$ an l.o. group and if $Inv_*(D)$ is l.o. what properties does D have?

Call an integral domain a $*$-PrĄfer domain if every nonzero finitely generated ideal of D is $*$-invertible. These domains were studied by Anderson, Anderson, Fontana and Zafrullah, in [1]. It was shown in [1] that D is a $*$-PrĄfer domain if and only if every nonzero two generated ideal of D is $*$-invertible.

Proposition 5.6. [1, Theorem 2.11] *An integral domain D is a $*$-PrĄfer domain if and only if $Inv_*(D)$ is a l.o. group with* $\inf(A, B) = (A + B)^*$ *for all $A, B \in Inv_*(D)$.*

Proof. Suppose that $Inv_*(D)$ is a lattice ordered group. Then for every pair $A, B \in Inv_*(D)$, $\inf(A, B) \in Inv_*(D)$. Now we note that as $(A + B)^* \supseteq A, B$ and if $C \in Inv_*(D)$ such that $C \supseteq A, B$ then $C \supseteq (A + B)^*$. Taking the duals we get $(A + B)^* = \inf(A, B)$. Now $(A + B)^* \in Inv_*(D)$ means, in particular that the sum of every two $*$-invertible $*$-ideals is a $*$-invertible ideal. Thus in particular the sum of every two nonzero integral principal ideals of D is $*$-invertible. This makes D a $*$-Prüfer domain. Conversely suppose that D is $*$-Prüfer. Then, by (vi) Theorem 2.2 of [1] the sum of every pair of $*$-invertible ideals is $*$-invertible. Thus for every pair $A, B \in Inv_*(D)$, $(A + B)^* \in Inv_*(D)$. Now from the previous considerations it is easy to see that for $A, B \in Inv_*(D)$, $(A + B)^* = \inf(A, B)$. □

Next let D be a $*$-PrĄfer domain for a star operation $*$ and suppose that $Inv_*(D)$ satisfies Conrad's F-condition. Let us translate Conrad's F-condition to ring-theoretic language for a $*$-PrĄfer domain. Note that $(Inv_*(D))^+$ is the set of integral $*$-invertible $*$-ideals of the $*$-PrĄfer domain D and $A \in Inv_*(D)$ being strictly positive means A is a proper integral ideal i.e. $A \subsetneq D$. Also $A, B \in Inv_*(D)$ being disjoint means $(A + B)^* = D$, we may call A, B $*$-comaximal. So, recalling that $\geq$ translates to $\subseteq$ in the ideal setup, Conrad's F-condition translates in $*$-PrĄfer domains to: Every proper integral $*$-invertible $*$-ideal is contained in at most a finite number of mutually $*$-comaximal proper integral $*$-invertible $*$-ideals. The notion of a basic element translates to a $*$-invertible $*$-ideal H such that every pair of proper $*$-invertible $*$-ideals containing H is comparable. Let us call the basic elements $H \in Inv_*(D)$ the basic ideals of D.

Now $Inv_*(D)$ is more interesting in case $*$ is of finite character. So we shall, for now, look into $Inv_*(D)$ for a star operation $*$ of finite character. In this case we know that a $*$-PrĄfer domain with $*$ of finite character is what is called a PrĄfer $*$-Multiplication Domain (P$*$MD). The beauty of this case is that each $A \in Inv_*(D)$ (and hence in $(Inv_*(D))^+$) is a $*$-ideal of finite type. So, every basic element of $A \in Inv_*(D)$, in this case is a $*$-ideal of finite type.

The case of $Inv_*(D)$, with $*$ of finite character, being lattice ordered satisfying Conrad's F-condition was first studied by Griffin in [14]. Recently it has been recalled in [27] to answer a question posed by Bazzoni in [4, 5]. Next, the case of $Inv_*(D)$, with $*$ of finite character, being a Riesz group satisfying Conrad's F-condition has been studied by Dumitrescu and Zafrullah [9] for the so-called t-Schreier domains. Recall that D is t-Schreier if $Inv_t(D)$ is a Riesz group.

We shall study the effect of Conrad's F-condition when, for $*$ of finite character, $Inv_*(D)$ is a pre-Riesz group and derive all the special cases of $Inv_*(D)$ being a Riesz group or an l.o. group. Using $*_s$ as a prototype of finite character star operations we shall study $Inv_{*_s}(D)$. Our treatment will be ab initio, as in the case of $G(D)$ pre-Riesz, and so new even for the known cases. Let us, temporarily, call a domain D a $*_s$-pre-Riesz domain if $Inv_{*_s}(D)$ is a pre-Riesz group.

First let us note that as in the case of pre-Riesz $G(D)$, an element $H \in Inv_{*_s}(D)$ will be homogeneous if $H \subsetneq D$ and given any pair $A, B \in Inv_{*_s}(D)$ with $H \subseteq A, B \subsetneq D$ we must have $(A, B)^{*_s} \neq D$. Next, as we have already seen, a pre-Riesz group satisfies Conrad's F-condition if and only if every strictly positive element exceeds at least one and at most a finite number of mutually disjoint homogeneous elements (cf. Corollary 3.9). Translating, a $*_s$-pre-Riesz domain satisfies Conrad's F-condition if and only if every proper $*_s$-invertible $*_s$-ideal of D is contained in at least one and at most a finite number of mutually $*_s$-comaximal homogeneous ideals of D. So, as in the PSP domain case we aim to link the homogeneous elements to the maximal $*_s$-ideals of $*_s$-pre-Riesz domains. But before that let us find a working characterization of $*_s$-pre-Riesz domains.

Proposition 5.7. *Let D be an integral domain and let $*$ be a star operation on D. Then the following are equivalent. (1) D is a $*_s$-pre-Riesz domain, (2) for all $A_1, A_2, ..., A_n \in Inv_{*_s}(D)^+$ with $(A_1, A_2, ..., A_n)^{*_s} \neq D$ there is a proper ideal $A \in Inv_{*_s}(D)^+$ such that $A_1, A_2, ..., A_n \subseteq A$. (3) for all $A_1, A_2, ..., A_n \in Inv_{*_s}(D)^+$ with $(A_1, A_2, ..., A_n)^* \neq D$ there is a proper $*_s$-invertible $*_s$-ideal A with $A_1, A_2, ..., A_n \subseteq A$. (4) for all $x_1, x_2, ..., x_n \in D\backslash\{0\}$ with $(x_1, x_2, ..., x_n)^* \neq D$ there is a proper $*_s$-invertible $*_s$-ideal A with $(x_1, x_2, ..., x_n) \subseteq A$.*

Proof. (1) $\Leftrightarrow$ (2) is obvious because D being a $*_s$-pre-Riesz domain means $Inv_{*_s}(D)$ is a pre-Riesz group and (2) is the definition of a pre-Riesz group. Next (2) $\Leftrightarrow$ (3) is direct because $A_1, A_2, ..., A_n$ are of finite type and so $(A_1, A_2, ..., A_n)^* = (A_1, A_2, ..., A_n)^{*_s}$. Also (3) $\Rightarrow$ (4) is obvious as $x_iD \in Inv_{*_s}(D)^+$, because every principal ideal is $*$-invertible for every $*$-operation. This leaves (4) $\Rightarrow$ (3). Let $A_1, A_2, ..., A_n \in Inv_{*_s}(D)^+$. Since each of A_i is a $*$-ideal of finite type there are finitely generated ideals B_i such that $A_i = B_i^* = (x_{1i}, x_{2i}, ..., x_{n_ii})^*$. Thus $(A_1, A_2, ..., A_n)^* = (\sum(x_{1i}, x_{2i}, ..., x_{n_ii})^*)^* = (\sum(x_{1i}, x_{2i}, ..., x_{n_ii}))^*$. Now $(A_1, A_2, ..., A_n)^* \neq D$ $\Rightarrow (\sum(x_{1i}, x_{2i}, ..., x_{n_ii}))^* \neq D$ and this, by (4) implies that there is a proper $*_s$-invertible $*_s$-ideal A with $\sum(x_{1i}, x_{2i}, ..., x_{n_ii}) \subseteq A$. But as A is a $*$-ideal we have $(\sum(x_{1i}, x_{2i}, ..., x_{n_ii}))^* \subseteq A$. But as $(A_1, A_2, ..., A_n)^* = (\sum(x_{1i}, x_{2i}, ..., x_{n_ii}))^*$ we have the result. □

Proposition 5.7 shows that $*_s$-pre-Riesz domains are nothing but the $*_s$-sub-PrĄfer domains studied in [8]. In [8] an indirect device was used to show that in a $*_s$-sub-PrĄfer domain the number of mutually disjoint homogeneous ideals containing a finitely generated ideal was linked to the number of maximal $*_s$-ideals containing that ideal. Here we aim to study, eventually, the effect of Conrad's F-condition on a $*_s$-sub-PrĄfer domain, though we look, enroute, into domains each of whose maximal $*_s$-ideal contains a homogeneous ideal. This requires a look into the nature of homogeneous elements.

Remark 5.8. *If H is a homogeneous element of a $*_s$-sub-PrĄfer domain D, then every proper $K \in Inv_{*_s}(D)^+$, with $H \subseteq K$, is homogeneous.*

Lemma 5.9. *Let H be a homogeneous ideal of a $*_s$-sub-PrÃ¼fer domain D. Then the following hold.*

(1) If ideals $H_1, H_2, ..., H_n$ are such that $H_i^{*_s}$ is proper of finite type for each i and all $H_i \supseteq H$ then there is a proper $K \in Inv_{*_s}(D)^+$ such that $K \supseteq H_i$ for each i. Moreover K is a homogeneous ideal.

(2) The set $P(H) = \{x \in D \backslash \{0\} : (x, H)^{*_s} \neq D\} \cup \{0\}$ is a maximal t-ideal of D.

Proof. (1) By Proposition 5.7 each of H_i is contained in some proper $J_i \in Inv_{*_s}(D)^+$. We claim that $(J_1, J_2, ..., J_n)^{*_s} \neq D$. For if $(J_1, J_2, ..., J_n)^{*_s} = D$ then by relabeling, if necessary, take $T = \{J_1, J_2, ..., J_r\}$ to be a maximal subset of $S = \{J_1, J_2, ..., J_n\}$ such that $J = (J_1, J_2, ..., J_r)^{*_s} \neq D$. Of course J must be contained in a proper $A \in Inv_{*_s}(D)^+$. But then for any $J_k \in S \backslash T$ we must have $(A, J_k)^{*_s} = D$. But this contradicts the fact that H is homogeneous. Thus $(J_1, J_2, ..., J_n)^{*_s} \neq D$ and there is a proper $K \in Inv_{*_s}(D)^+$ such that $K \supseteq J_i \supseteq H_i$, for each i. That K is homogeneous follows directly from the definition of homogeneous.

(2) We first establish that if $x, y \in P(H)$ then $x + y \in P(H)$. For this note that $(x, H)^{*_s} \neq D$ and $(y, H)^{*_s} \neq D$. So there exist proper $H_1, H_2 \in Inv_{*_s}(D)^+$ such that $H_1 \supseteq (x, H)$ and $H_2 \supseteq (y, H)$. Now both H_1, H_2 are proper and contain H, there must be a proper $K \in Inv_{*_s}(D)^+$ such that K contains H_1, H_2, hence x, y, H and hence $x + y$ and H. Thus $x + y \in P(H)$. Indeed if $(x, H)^{*_s} \neq D$, then for each $r \in R$ we have $(rx, H)^{*_s} \neq D$. To see that $P(H)$ is a $*_s$-ideal let $x_1, x_2, ..., x_n \in P(H)$. Then $H_i = (x_i, H)^{*_s}$ are proper of finite type and contain H so, by (1), there is a proper $K \in Inv_{*_s}(D)^+$ such that $(H_1, H_2, ..., H_n) \subseteq K$. Now $(x_1, x_2, ..., x_n)^* \subseteq (H_1, H_2, ..., H_n)^* \subseteq K$ and $K \subseteq P(H)$, because $(K, H)^{*_s} = K \neq D$. To see that $P(H)$ is a maximal $*_s$-ideal note that $x \notin P(H)$ implies that $(x, H)^{*_s} = D$. Hence there can be no prime $*_s$-ideal properly containing $P(H)$. □

Remark 5.10. *If for H a homogeneous ideal $M = P(H)$ then every proper J with $J \supseteq H \in Inv_{*_s}(D)^+$, must be contained in M. Also if M is a maximal $*_s$-ideal containing a homogeneous ideal H, then $M = P(H)$.*

Lemma 5.11. *Let D be a $*_s$-sub-PrÃ¼fer domain. Then $H \in Inv_{*_s}(D)^+$ is a homogeneous ideal if and only if H is contained in a unique maximal $*_s$-ideal.*

Proof. If $H \in Inv_{*_s}(D)^+$ is homogeneous then there is a maximal ideal $P(H)$ containing H. Suppose that there is another maximal $*_s$-ideal M containing H then since for each $x \in M$, $(x, H)^{*_s} \subseteq M$, $(x, H)^{*_s} \neq D$. But then $x \in P(H)$. So $M \subseteq P(H)$. Since M is assumed to be a maximal $*_s$-ideal we must have $M = P(H)$. Conversely, suppose that H is contained in a unique maximal $*_s$-ideal M. Then for all proper $A, B \in Inv_{*_s}(D)^+$ with $A, B \supseteq H$ then $A, B \subseteq M$, because M is the only maximal $*_s$-ideal containing H, but then $(A, B)^{*_s} \subseteq M$. □

We now characterize the $*_s$-sub-PrÃ¼fer domains with maximal $*_s$-ideals associated to homogeneous elements.

Proposition 5.12. *Let D be a $*_s$-sub-PrÃ¼fer domain. If every maximal $*_s$-ideal of D contains a homogeneous ideal then for every ideal I with I^{*_s} proper and of finite type is contained in a homogeneous ideal of D. Conversely if every ideal I with I^{*_s} proper and of finite type is contained in a homogeneous ideal of D then every maximal $*_s$-ideal contains a homogeneous ideal provided* *that the following condition holds: if a maximal $*_s$-ideal M is contained in a union of any family $\{M_\alpha\}$ of maximal $*_s$-ideals then $M = M_\alpha$ for some α.*

Proof. Suppose that D is $*_s$-sub-Prüfer and that every maximal $*_s$-ideal contains a homogeneous ideal. Take an ideal I with I^{*_s} proper and of finite type. Then I^{*_s} and hence I must be contained in a maximal $*_s$-ideal $M = P(H)$ where H is homogeneous. Then $(I, H)^{*_s} \subseteq P(H)$ and so $(I, H)^{*_s} \neq D$. Thus by Proposition 5.7, there is a proper $H_1 \in Inv_{*_s}(D)^+$ such that $H_1 \supseteq I, H$. But as H_1 contains H, H_1 is homogeneous. That H_1 is contained in $P(H)$ follows from the fact that for each $x \in H_1$ we have $(x, H) \subseteq H_1$ and so $(x, H)^{*_s} \subseteq H_1 \neq D$.

Conversely suppose that every ideal I with I^{*_s} proper and of finite type there is a homogeneous ideal containing I. Let M be a maximal $*_s$-ideal containing I and let $\{P(H)\}$ be the set of all the maximal $*_s$-ideals each associated with a homogeneous ideal. Because, by the condition, for each $x \in M$, $x \in H$ for some homogeneous ideal we conclude that $M \subseteq \cup P(H)$. We conclude by the condition that $M = P(H)$ for some homogeneous H. □

Theorem 5.13. *Let D be a $*_s$-sub-PrĄfer domain. Then $Inv_{*_s}(D)$ satisfies Conrad's F-condition if and only if every nonzero nonunit x of D belongs to at most a finite number of maximal $*_s$-ideals of D.*

Proof. Recall from Corollary 3.9, that the pre-Riesz group $Inv_{*_s}(D)$ satisfies Conrad's F-condition if and only if every strictly positive element (proper integral $*_s$-invertible $*_s$-ideal of D) exceeds (is contained in) at least one and at most a finite number of mutually disjoint homogeneous elements of $Inv_{*_s}(D)$ (ideals of D). So if $Inv_{*_s}(D)$ satisfies Conrad's F-condition then every proper integral $*_s$-invertible $*_s$-ideal, say A, is contained in at least one homogeneous ideal H and hence at least one maximal $*_s$-ideal $P(H)$ and at most a finite number $P(H_1), P(H_2), ..., P(H_n)$ of maximal $*_s$-ideals associated with homogeneous ideals. Suppose that n is the largest such number. We show that $P(H_1), P(H_2), ..., P(H_n)$ are the only maximal $*_s$-ideals containing A. For this let M be a maximal $*_s$-ideal containing A and that $M \neq P(H_i)$, $i = 1, ..., n$. Then there is $x \in M \backslash \cup P(H_i)$, $i = 1, ..., n$. Now $(x, A)^{*_s}$ is of finite type and contained in M. Because D is $*_s$-sub-Prüfer there is a $*_s$-invertible proper integral $*_s$-ideal K containing $(x, A)^{*_s}$ and because of the F-condition there is a homogeneous ideal L containing $K \supseteq A$. But then there is one more $*_s$-maximal ideal $P(L)$, associated with a homogeneous ideal, that contains A, a contradiction. Since for every nonzero nonunit $x \in D$, xD is a proper $*_s$-invertible $*_s$-ideal we have the result.

Conversely suppose that every nonzero nonunit of D is contained in at most a finite number of maximal $*_s$-ideals. Let M be a maximal $*_s$-ideal and let $x \in M\backslash\{0\}$. Let $\{M, M_1, M_2, ..., M_n\}$ be the set of all the maximal $*_s$-ideals of D containing x. Since $M \neq M_i$ there are $x_i \in M\backslash M_i$. Set $A = (x, x_1, x_2, ..., x_n)$. Clearly $A^{*_s} \subseteq M$ and by the construction of A, M is the only maximal $*_s$-ideal containing A^{*_s}. Also $A^{*_s} \neq D$ and D being $*_s$-sub-Prüfer there is a proper $H \in Inv_{*_s}(D)^+$ such that $A^{*_s} \subseteq H$. Now H is a $*_s$-invertible $*_s$-ideal contained in a unique maximal $*_s$-ideal M, H is homogeneous. Thus every maximal $*_s$-ideal of D contains a homogeneous ideal and hence every maximal $*_s$-ideal is associated to a homogeneous ideal. Next as every principal ideal is contained in at most a finite number of maximal $*_s$-ideals implies that every $*_s$-invertible $*_s$-ideal, being of finite type, is contained in at most a finite number of maximal $*_s$-ideals. Next as every maximal $*_s$-ideal $M = P(H)$ for some homogeneous ideal H, for a $*_s$-invertible $*_s$-ideal

$A \subseteq P(H)$ we have $(A, H)^{*_s} \subseteq P(H)$ and so by the $*_s$-sub-Prüfer property there is a $*_s$-invertible $*_s$-ideal $K \supseteq (A, H)^{*_s}$ and this K is obviously homogeneous. Thus A being contained in at most a finite number of maximal $*_s$- ideals translates into A being contained in at most a finite number of mutually $*_s$-comaximal homogeneous ideals. □

As we have already mentioned, an integral domain D is called a PrĄfer $*$-multiplication domain (P$*$MD) if $*$ is of finite character and if every nonzero finitely generated ideal of D is $*$-invertible. According to Griffin [14] D is a PVMD (i. e., $* = t$) if and only if D_M is a valuation domain for each $M \in t - \max(D)$. Using similar arguments one can show that for a star operation $*$ of finite character, D is a P$*$MD if and only if D_M is a valuation domain for every maximal $*$-ideal M [15]. Repeating "star operation of finite character" over and over again may be a bit cumbersome. So as before we resort to the convention of using $*_s$, for any star operation $*$, as the prototype of a star operation of finite character we shall call D a P$*$MD if every nonzero finitely generated ideal of D is $*_s$-invertible. So by a P$*$MD we shall mean a $*_s$-PrĄfer domain. Now by Proposition 5.6, D is $*_s$-PrĄfer if and only if $Inv_{*_s}(D)$ is an l.o. group. Similarly we can call a domain a $*$-Schreier domain if $Inv_{*_s}(D)$ is a Riesz group. Now as a pre-Riesz group is a generalization of both the Riesz and l.o. groups the results proved for pre-Riesz groups also hold for both Riesz and l.o. groups. In the same vein results proved for $*_s$-sub-PrĄfer domains hold for both $*_s$-Schreier domains and $*_s$-PrĄfer domains.

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