# SOME REMARKS ON PRÜFER $\star$–MULTIPLICATION DOMAINS AND CLASS GROUPS

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Abstract. Let $D$ be an integral domain with quotient field $K$ and let $X$ be an indeterminate over $D$. Also, let $\boldsymbol{\mathcal{T}} := \{T_\lambda \mid \lambda \in \Lambda\}$ be a defining family of quotient rings of $D$ and suppose that $*$ is a finite type star operation on $D$ induced by $\boldsymbol{\mathcal{T}}$. We show that $D$ is a P$*$MD (resp., P$v$MD) if and only if $(\boldsymbol{c}_D(fg))^* = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^*$ (resp., $(\boldsymbol{c}_D(fg))^w = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^w$) for all $0 \neq f, g \in K[X]$. A more general version of this result is given in the semistar operation setting. We give a method for recognizing P$v$MD's which are not P$*$MD's for a certain finite type star operation $*$. We study domains $D$ for which the $*$–class group $\mathrm{Cl}^*(D)$ equals the $t$–class group $\mathrm{Cl}^t(D)$ for any finite type star operation $*$, and we indicate examples of P$v$MD's $D$ such that $\mathrm{Cl}^*(D) \subsetneq \mathrm{Cl}^t(D)$. We also compute $\mathrm{Cl}^v(D)$ for certain valuation domains $D$.

## Introduction and Background

Let $D$ be an integral domain with quotient field $K$. Let $\overline{\boldsymbol{F}}(D)$ be the set of all nonzero $D$–submodules of $K$ and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of $D$, i.e., $E \in \boldsymbol{F}(D)$ if $E \in \overline{\boldsymbol{F}}(D)$ and there exists a $0 \neq d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated $D$–submodules of $K$. Then, obviously $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

A *semistar operation* on $D$ is a map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D), E \mapsto E^\star$, such that the following properties hold for all $0 \neq x \in K$ and all $E, F \in \overline{\boldsymbol{F}}(D)$:

($\star_1$) $(xE)^\star = xE^\star$;
($\star_2$) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

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Given a semistar operation $\star$ on $D$, the following basic formulas, which hold for all $E, F \in \overline{\boldsymbol{F}}(D)$, follow easily from the axioms:

$$\begin{aligned}
(EF)^\star &= (E^\star F)^\star = (EF^\star)^\star = (E^\star F^\star)^\star \ ; \\
(E+F)^\star &= (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star \ ; \\
(E:F)^\star &\subseteq (E^\star : F^\star) = (E^\star : F) = (E^\star : F)^\star \,, \ \text{if } \ (E:F) \neq (0)\,; \\
(E \cap F)^\star &\subseteq E^\star \cap F^\star = (E^\star \cap F^\star)^\star \,.
\end{aligned}$$

(Cf. for instance [22, Theorem 1.2 and p. 174].)

A *(semi)star operation* is a semistar operation which when restricted to $\boldsymbol{F}(D)$ is a star operation (the reader may consult [31, Sections 32 and 34] for a quick review of star operations, which are denoted by the symbol $\ast$). It is easy to see that a semistar operation $\star$ on $D$ is a (semi)star operation if and only if $D^\star = D$.

If $\star$ is a semistar operation on $D$, then there is a map $\star_{_f} : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$ defined as follows:

$$E^{\star_f} := \bigcup \{ F^\star \mid \ F \in \boldsymbol{f}(D) \text{ and } F \subseteq E \} \qquad \text{for all } E \in \overline{\boldsymbol{F}}(D).$$

It is easy to see that $\star_{_f}$ is a semistar operation on $D$, called the *semistar operation of finite type associated to* $\star$. Note that $F^\star = F^{\star_f}$ for all $F \in \boldsymbol{f}(D)$. A semistar operation $\star$ is called a *semistar operation of finite type* (or a *semistar operation of finite character*) if $\star = \star_{_f}$. It is easy to see that $(\star_{_f})_{_f} = \star_{_f}$ (i.e., $\star_{_f}$ is of finite type).

If $\star_1$ and $\star_2$ are two semistar operations on $D$, we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for all $E \in \overline{\boldsymbol{F}}(D)$. This is equivalent to saying that $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$ for all $E \in \overline{\boldsymbol{F}}(D)$. Obviously, for any semistar operation $\star$ on $D$, we have $\star_{_f} \leq \star$, and if $\star_1 \leq \star_2$, then $(\star_1)_{_f} \leq (\star_2)_{_f}$.

Let $I \subseteq D$ be a nonzero ideal of $D$ and let $\star$ be a semistar operation on $D$. We say that $I$ is a *quasi–$\star$–ideal* (resp., *$\star$–ideal*) of $D$ if $I^\star \cap D = I$ (resp., $I^\star = I$). Similarly, we call a quasi–$\star$–ideal (resp., $\star$–ideal) of $D$ a *quasi–$\star$–prime* (resp., *$\star$–prime*) *ideal* of $D$ if it is also a prime ideal. We call a maximal element in the set of all proper quasi–$\star$–ideals (resp., $\star$–ideals) of $D$ a *quasi–$\star$–maximal* (resp., *$\star$–maximal*) *ideal* of $D$. Note that if $I \subseteq D$ is a $\star$–ideal, then it is also a quasi–$\star$–ideal, and when $D = D^\star$ (i.e., when $\star$ is a(semi)star operation), the notions of quasi–$\star$–ideal and $\star$–ideal coincide.

It is not hard to prove that a quasi–$\star$–maximal ideal is a prime ideal and that each proper quasi–$\star_{_f}$–ideal is contained in a quasi–$\star_{_f}$–maximal ideal. More details can be found in [25, page 4781]. We will denote the set of quasi–$\star$–prime (resp., $\star$–prime) ideals of $D$ by $\operatorname{QSpec}^\star(D)$ (resp., $\operatorname{Spec}^\star(D)$) and the set of quasi–$\star$–maximal (resp., $\star$–maximal) ideals of $D$ by $\operatorname{QMax}^\star(D)$ (resp., $\operatorname{Max}^\star(D)$). By the previous observations, we have that $\operatorname{QMax}^{\star_f}(D)$ (resp., $\operatorname{Max}^{\star_f}(D)$) is non-empty for each semistar (resp., (semi)star) operation $\star$ on $D$.

If $T$ is an overring of $D$, then we can define a semistar operation $\star_{\{T\}}$ on $D$ by $E^{\star_{\{T\}}} := ET$ for all $E \in \overline{\boldsymbol{F}}(D)$. It is easily seen that $\star_{\{T\}}$ is a semistar (non (semi)star, if $D \subsetneq T$) operation of finite type on $D$.

If $\{\star_\lambda \mid \lambda \in \Lambda\}$ is a family of semistar operations on $D$, then $\wedge\{\star_\lambda \mid \lambda \in \Lambda\}$ is the semistar operation on $D$ defined as follows:

$$E^{\wedge\{\star_\lambda \mid \lambda\in\Lambda\}} := \bigcap\{E^{\star_\lambda} \mid \lambda \in \Lambda\} \quad \text{for all } E \in \overline{\boldsymbol{F}}(D).$$

In particular, if $\boldsymbol{\mathcal{T}} := \{T_\lambda \mid \lambda \in \Lambda\}$ is a given family of overrings of $D$, then $\wedge_{\boldsymbol{\mathcal{T}}}$ denotes the semistar operation $\wedge\{\star_{\{T_\lambda\}} \mid \lambda \in \Lambda\}$.

Let $\Delta$ be a set of prime ideals of an integral domain $D$ and $\boldsymbol{\mathcal{L}}(\Delta)$ the set of localizations $\{D_P \mid P \in \Delta\}$. The semistar operation $\star_\Delta := \wedge_{\boldsymbol{\mathcal{L}}(\Delta)}$ is called the *spectral semistar operation associated to* $\Delta$. A semistar operation $\star$ on an integral domain $D$ is called a *spectral semistar operation* if there exists a subset $\Delta$ of the prime spectrum $\mathrm{Spec}(D)$ of $D$ such that $\star = \star_\Delta$. Note that for $\Delta = \emptyset$, we set $\star_\Delta := \star_{\{K\}}$, where $K$ is the quotient field of $D$.

When $\Delta := \Delta(\star_{_f}) := \mathrm{QMax}^{\star_f}(D)$, we set $\widetilde{\star} := \star_{\Delta(\star_{_f})}$, i.e.,

$$E^{\widetilde{\star}} := \bigcap\left\{ED_M \mid M \in \mathrm{QMax}^{\star_f}(D)\right\} \quad \text{for all } E \in \overline{\boldsymbol{F}}(D).$$

A semistar operation $\star$ is said to be *stable* if $(E \cap F)^\star = E^\star \cap F^\star$ for all $E, F \in \overline{\boldsymbol{F}}(D)$. Note that if $T$ is an overring of an integral domain $D$, then $\star_{\{T\}}$ is stable if and only if $T$ is $D$–flat (cf. [50, Proposition 1.7] and [42, Theorem 7.4(i)]). Clearly, if $\{\star_\lambda \mid \lambda \in \Lambda\}$ is a family of stable semistar operations on $D$, then $\wedge\{\star_\lambda \mid \lambda \in \Lambda\}$ is also a stable semistar operation on $D$. In particular, if $\boldsymbol{\mathcal{T}}$ is a family of flat overrings of $D$, then $\wedge_{\boldsymbol{\mathcal{T}}}$ is a stable semistar operation on $D$. Thus every spectral semistar operation is stable (cf. also [22, Lemma 4.1(3)]).

It is well known that the semistar operation $\widetilde{\star}$ is a stable semistar operation of finite type [22, Corollaries 3.9 and 4.6]. We call $\widetilde{\star}$ the *stable semistar operation of finite type associated to* $\star$. Furthermore, it is not hard to prove that $\mathrm{QMax}^{\widetilde{\star}}(D) = \mathrm{QMax}^{\star_f}(D)$ [25, Corollary 3.5(2)]; thus $\widetilde{\widetilde{\star}} = \widetilde{\star} = \widetilde{\star_{_f}}$. Clearly $\widetilde{\star} \leq \star$, and since $(\widetilde{\star})_{_f} = \widetilde{\star}$, then $\widetilde{\star} \leq \star_{_f} \leq \star$. Moreover, it is known that if $\star_1 \leq \star_2$, then $\widetilde{\star_1} \leq \widetilde{\star_2}$ [25, Propositions 3.1 and 3.4(3)].

For each $E \in \overline{\boldsymbol{F}}(D)$, set $E^{-1} := (D : E) := \{z \in K \mid zE \subseteq D\}$. Clearly $E \in \overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$ if and only if $E^{-1} = \{0\}$. As usual, we let $v_D$ (or just $v$) denote the $v$–(semi)star operation defined by $E^v := (D : (D : E)) = \left(E^{-1}\right)^{-1}$ for all $E \in \overline{\boldsymbol{F}}(D)$. (Note that $E \in \overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$ implies that $E^v = K$.) We denote $(v_D)_{_f}$ by $t_D$ (or just by $t$), the $t$–(semi)star operation on $D$; and we denote the stable semistar operation of finite type associated to $v_D$ (or, equivalently, to $t_D$) by $w_D$ (or just by $w$), i.e., $w_D := \widetilde{v_D} = \widetilde{t_D}$. Clearly $w_D \leq t_D \leq v_D$. Moreover, from [31, Theorem 34.1(4)], we immediately deduce that $\star \leq v_D$, and thus $\widetilde{\star} \leq w_D$ and $\star_{_f} \leq t_D$, for each (semi)star operation $\star$ on $D$.

**Remark.** Note that the (semi)star operation $\widetilde{v}$ coincides with the (semi)star operation defined as follows:

$$E^w := \bigcup\{(E : H) \mid H \in \boldsymbol{f}(D) \text{ and } H^v = D\} \quad \text{for all } E \in \overline{\boldsymbol{F}}(D)\,.$$

In the "star operation setting", this operation was first considered by J. Hedstrom and E. Houston in 1980 [35, Section 3] under the name of the $\mathrm{F}_\infty$–operation. Later, from 1997, this operation was intensively studied by F. Wang and R. McCasland (cf. [51] and [52]) under the name of the *$w$–operation.* Also note that the notion of $w$–ideal coincides with the notion of semi-divisorial ideal considered by S. Glaz and W. Vasconcelos in 1977 [32]. Finally, in 2000, for each star operation $*$ on $D$, D.D. Anderson and S.J. Cook [7] considered the star operation $*_w$ on $D$ defined as follows:

$$E^{*_w} := \bigcup \{(E : H) \mid H \in \boldsymbol{f}(D) \text{ and } H^* = D\} \quad \text{for all } E \in \boldsymbol{F}(D)\,.$$

It can be shown that when $\star = *$ is a star operation, then $*_w$ coincides with $\widetilde{*}$ (defined in the obvious way as a star operation on $\boldsymbol{F}(D)$) [7, Corollary 2.10].

Finally, note that a deep link between the semistar operations of type $\widetilde{\star}$ and localizing systems of ideals was established by M. Fontana and J. Huckaba in [22].

Let $\star$ be a semistar operation on the integral domain $D$.

For $I \in \overline{\boldsymbol{F}}(D)$, we say that $I$ is *$\star$–finite* if there exists a $J \in \boldsymbol{f}(D)$ such that $J^\star = I^\star$. (Note that in the above definition, we do not require that $J \subseteq I$.) It is immediate to see that if $\star_1 \leq \star_2$ are semistar operations and $I$ is $\star_1$–finite, then $I$ is $\star_2$ –finite. In particular, if $I$ is $\star_{_f}$–finite, then it is $\star$–finite. The converse is not true in general, and one can prove that $I$ is $\star_{_f}$–finite if and only if there exists $J \in \boldsymbol{f}(D)$, $J \subseteq I$, such that $J^\star = I^\star$ [28, Lemma 2.3]. This result was proved in the star operation setting by M. Zafrullah in [58, Theorem 1.1].

For $I$ a nonzero ideal of $D$, we say that $I$ is *$\star$–invertible* if $(II^{-1})^\star = D^\star$. From the fact that $\mathrm{QMax}^{\widetilde{\star}}(D) = \mathrm{QMax}^{\star_{_f}}(D)$, it easily follows that an ideal $I$ is $\widetilde{\star}$–invertible if and only if $I$ is $\star_{_f}$–invertible (note that if $\star$ is a semistar operation of finite type, then $(II^{-1})^\star = D^\star$ if and only if $II^{-1} \not\subseteq M$ for all $M \in \mathrm{QMax}^\star(D)$). It is well known that if $I$ is $\star_{_f}$–invertible, then $I$ and $I^{-1}$ are both $\star_{_f}$–finite [28, Proposition 2.6].

An integral domain $D$ is called a *Prüfer $\star$–multiplication domain* (for short, *P$\star$MD*) if every nonzero finitely generated ideal of $D$ is $\star_{_f}$–invertible (cf. for instance [24]). Note that for $\star = *$ a star operation of finite type on $D$, P$*$MD's were introduced by Houston, Malik, and Mott in [37] as $*$–multiplication domains (for short, $*$–MD's). When $\star = v$, we have the classical notion of P$v$MD (cf. for instance [33], [44] and [40]); when $\star = d$, where $d$ denotes the identity (semi)star operation, we have the notion of Prüfer domain [31, Theorem 22.1]. Note that from the definition and from the previous observations, it immediately follows that the notions of P$\star$MD, P$\star_{_f}$MD, and P$\widetilde{\star}$MD coincide.

Let $K$ be the quotient field of an integral domain $D$ and let $X$ be an indeterminate over $K$. For each $0 \neq h \in K[X]$, we denote by $\boldsymbol{c}_D(h)$ the *content of $h$ with respect to $D$,* i.e., the $D$–submodule of $K$ generated by the coefficients of $h$. Clearly $\boldsymbol{c}_D(h) \in \boldsymbol{f}(D)$, and if $T$ is an overring of $D$, then $\boldsymbol{c}_T(h) = \boldsymbol{c}_D(h)T$.

Gauss' Lemma for the content of polynomials holds for Dedekind domains (or, more generally, for Prüfer domains). A more precise statement is the following:

**Gauss-Gilmer-Tsang Theorem** [31, Corollary 28.5]. *Let $D$ be an integral domain with quotient field $K$. Then $D$ is a Prüfer domain if and only if $\boldsymbol{c}_D(fg) = \boldsymbol{c}_D(f)\boldsymbol{c}_D(g)$ for all $0 \neq f, g \in K[X]$.*

**Remark.** W. Krull [41, page 557] showed that if $D$ is an integrally closed domain with quotient field $K$, then we have $(\boldsymbol{c}_D(fg))^v = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^v$ for all $0 \neq f, g \in K[X]$, and called it Gauss' Theorem. Obviously the currently known Gauss' Lemma (that goes as: the product of two primitive polynomials over a UFD is again primitive [17, page 165]) and Gauss' own statement (let $f$ and $g$ be monic polynomials in one indeterminate with rational coefficients, if the coefficients of $f$ and $g$ are not all integers, then the coefficients of $fg$ are not all integers [19, page 1]) follow from Krull's above-mentioned result. As pointed out before the statement of the above theorem, Krull's Gauss' Theorem also holds for Prüfer domains; because for $D$ a Prüfer domain with quotient field $K$, we have $(\boldsymbol{c}_D(f))^v = \boldsymbol{c}_D(f)$ for all $0 \neq f \in K[X]$. Thus Krull's Gauss' Theorem for Prüfer domains becomes: if $D$ is a Prüfer domain with quotient field $K$, then $\boldsymbol{c}_D(fg) = \boldsymbol{c}_D(f)\boldsymbol{c}_D(g)$ for all $0 \neq f, g \in K[X]$. The converse of this statement was included in H. Tsang's unpublished dissertation [49]. This result was later, and independently, rediscovered by R. Gilmer and published in [30]. Since neither of these authors attributed their result to Gauss, we feel it appropriate to include their names with Gauss' name. For more on the history of Gauss' Lemma, the reader may consult Anderson [3].

For general integral domains, we always have the inclusion of ideals $\boldsymbol{c}_D(fg) \subseteq \boldsymbol{c}_D(f)\boldsymbol{c}_D(g)$, and more precisely we have the following:

**Dedekind–Mertens Lemma** [31, Theorem 28.1]. *Let $0 \neq f, g \in K[X]$ and let $m := \deg(g)$. Then*

$$\boldsymbol{c}_D(f)^m \boldsymbol{c}_D(fg) = \boldsymbol{c}_D(f)^{m+1}\boldsymbol{c}_D(g)\,.$$

In Section 1, we prove a semistar extension of the Gauss-Gilmer-Tsang Theorem (as stated above), i.e., we show that if $\star$ is a stable semistar operation of finite type defined on an integral domain $D$, then $D$ is a P$\star$MD if and only if $\boldsymbol{c}_D(fg)^\star = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star$ for all $0 \neq f, g \in K[X]$. Using this result, we show that there is an abundance of P$v$MD's which are not P$\star$MD's for appropriate stable (semi)star operations $\star$ of finite type on $D$.

For a finite type star operation $*$ on $D$, let $\mathrm{Inv}^*(D)$ be the group of $*$–invertible $*$–ideals of $D$ under $*$–multiplication and let $\mathrm{Prin}(D)$ be the subgroup of nonzero principal fractional ideals of $D$. Call $\mathrm{Cl}^*(D) := \mathrm{Inv}^*(D)/\mathrm{Prin}(D)$ the $*$–class group of $D$. The $*$–class groups were discussed in [10].

In Section 2, we study the $*$–class group and identify a situation in which for every finite type star operation $*$ on $D$, we have $\mathrm{Cl}^*(D) = \mathrm{Cl}^t(D)$; and using the results of Section 1, we give examples of integral domains $D$ for which $\mathrm{Cl}^*(D) \subsetneq \mathrm{Cl}^t(D)$ for some finite type star operation $*$ on $D$.

In Section 3, we deepen the study of the $v$–class group with special attention to the case of valuation domains. In particular, we compute $\mathrm{Cl}^v(D)$ when $D$ is a valuation domain with branched maximal ideal.

## 1. Prüfer $\star$–multiplication domains

With all the introduction at hand, we start right away with the promised characterization of P$\star$MD's. Using Theorem 1.1 below, we conclude that $D$ is a P$v$MD if and only if $\boldsymbol{c}_D(fg)^w = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^w$ for all $0 \neq f, g \in K[X]$. Also, using the proof of Theorem 1.1, we give a method for recognizing a P$v$MD which has a stable (semi)star operation $\star$ of finite type such that $D$ is not a P$\star$MD.

**Theorem 1.1.** *Let $D$ be an integral domain with quotient field $K$, let $X$ be an indeterminate over $K$, and let $\star$ be a stable semistar operation of finite type defined on $D$. Then the following are equivalent:*

(i) $\boldsymbol{c}_D(fg)^\star = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star$ *for all* $0 \neq f, g \in K[X]$.
(ii) $D_M$ *is a valuation domain for all* $M \in \mathrm{QMax}^\star(D)$.
(iii) *$D$ is a P$\star$MD.*

*Proof.* By the observations in the previous section, we know that under the present hypotheses, $\star = \widetilde{\star}$ [22, Corollary 3.9(2)], and thus $F^\star D_M = FD_M$ for all $M \in \mathrm{QMax}^\star(D)$ and $F \in \boldsymbol{f}(D)$.

(i)$\Rightarrow$(ii) Let $M \in \mathrm{QMax}^\star(D)$. From (i), we deduce that $\boldsymbol{c}_{D_M}(fg) = \boldsymbol{c}_D(fg)D_M = \boldsymbol{c}_D(fg)^\star D_M = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star D_M = \boldsymbol{c}_D(f)\boldsymbol{c}_D(g)D_M = \boldsymbol{c}_{D_M}(f)\boldsymbol{c}_{D_M}(g)$. This implies that $D_M$ is a valuation domain (i.e., a local Prüfer domain) by the Gauss-Gilmer-Tsang Theorem.

(ii)$\Rightarrow$(iii) Let $F \in \boldsymbol{f}(D)$. Note that for each flat overring $T$ of $D$, we have $F^{-1}T = (FT)^{-1}$. Also recall that for all $M \in \mathrm{QMax}^\star(D)$, every nonzero finitely generated ideal is invertible in the valuation domain $D_M$. Therefore, we have that $(FF^{-1})^\star = \bigcap\{(FF^{-1})D_M \mid M \in \mathrm{QMax}^\star(D)\} = \bigcap\{FD_M F^{-1}D_M \mid M \in \mathrm{QMax}^\star(D)\} = \bigcap\{(FD_M(FD_M)^{-1} \mid M \in \mathrm{QMax}^\star(D)\} = \bigcap\{D_M \mid M \in \mathrm{QMax}^\star(D)\} = D^\star$.

(iii)$\Rightarrow$(i) By the Dedekind-Mertens Lemma, $\boldsymbol{c}_D(f)^m\boldsymbol{c}_D(fg) = \boldsymbol{c}_D(f)^{m+1}\boldsymbol{c}_D(g)$ for all $0 \neq f, g \in K[X]$, where $m = \deg(g)$. In particular, we have $(\boldsymbol{c}_D(f)^m\boldsymbol{c}_D(fg))^\star = (\boldsymbol{c}_D(f)^{m+1}\boldsymbol{c}_D(g))^\star$. Since $D$ is a P$\star$MD, if $F := \boldsymbol{c}_D(f) \in \boldsymbol{f}(D)$, then $(FF^{-1})^\star = D^\star = (F^m(F^m)^{-1})^\star$. Therefore:

$$\begin{aligned}
\boldsymbol{c}_D(fg)^\star &= ((F^m(F^m)^{-1})^\star\boldsymbol{c}_D(fg))^\star \\
&= (\boldsymbol{c}_D(f)^m(F^m)^{-1}\boldsymbol{c}_D(fg))^\star = (\boldsymbol{c}_D(f)^{m+1}(F^m)^{-1}\boldsymbol{c}_D(g))^\star \\
&= (F^m(F^m)^{-1}\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star = ((F^m(F^m)^{-1})^\star\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star \\
&= (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star.
\end{aligned}$$

□

**Corollary 1.2.** *Let $D$ be an integral domain with quotient field $K$, let $X$ be an indeterminate over $K$, and let $\star$ be a semistar operation defined on $D$. Then the following are equivalent:*

(i) $\boldsymbol{c}_D(fg)^{\widetilde{\star}} = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^{\widetilde{\star}}$ *for all* $0 \neq f, g \in K[X]$.
(ii) $D_M$ *is a valuation domain for all* $M \in \mathrm{QMax}^{\star_f}(D)$.
(iii) *$D$ is a P$\star$MD.*

*Proof.* Apply Theorem 1.1 to $\widetilde{\star}$, the stable semistar operation of finite type associated to $\star$. Recall that from $\mathrm{QMax}^{\star_f}(D) = \mathrm{QMax}^{\widetilde{\star}}(D)$, we already deduced that the notions of P$\star$MD and P$\widetilde{\star}$MD coincide. □

**Corollary 1.3.** *Let $D$ be an integral domain and let $\star$ be a semistar operation of finite type induced by a family $\boldsymbol{\mathcal{T}}$ of flat overrings of $D$, i.e., $\star = \wedge_{\boldsymbol{\mathcal{T}}}$. Then $D$ is a P$\star$MD if and only if $\boldsymbol{c}_T(fg) = \boldsymbol{c}_T(f)\boldsymbol{c}_T(g)$ for all $0 \neq f, g \in K[X]$ (i.e., $T$ is a Prüfer domain) and all $T \in \boldsymbol{\mathcal{T}}$.*

*Proof.* Note that in this case, $\star$ is stable because each overring $T \in \boldsymbol{\mathcal{T}}$ is flat, and $\star$ is of finite type by assumption. Therefore $\star = \widetilde{\star}$. Moreover, we have $\boldsymbol{c}_D(h)^\star T = \boldsymbol{c}_D(h)T = \boldsymbol{c}_T(h)$ for all $0 \neq h \in K[X]$. The conclusion then follows immediately from Theorem 1.1 since $\boldsymbol{c}_D(fg)^\star = \bigcap\{\boldsymbol{c}_T(fg) \mid T \in \boldsymbol{\mathcal{T}}\}$ and $(\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star = \bigcap\{\boldsymbol{c}_T(f)\boldsymbol{c}_T(g) \mid T \in \boldsymbol{\mathcal{T}}\}$. □

**Remark 1.4.** Let $D$ be an integral domain, $\Delta$ a subset of $\mathrm{Spec}(D)$, and $\star := \star_\Delta$, the spectral semistar operation associated to $\Delta$. If we assume that $\Delta$ is quasi-compact (as a subspace of $\mathrm{Spec}(D)$ endowed with the Zariski topology), then the semistar operation $\star$ is a stable semistar operation of finite type [22, Corollary 4.6(2)], and thus we can apply Corollary 1.3 to this case.

The next corollary is a particularly significant case of Corollary 1.3.

**Corollary 1.5.** *Let $D$ be an integral domain with quotient field $K$, let $X$ be an indeterminate over $K$, and let $\star$ be the (semi)star operation of finite type induced by a defining family $\boldsymbol{\mathcal{T}}$ of $D$ consisting of quotient rings of $D$, i.e., $\boldsymbol{\mathcal{T}} := \{T_\lambda \mid \lambda \in \Lambda\}$ with $D = \bigcap\{T_\lambda \mid \lambda \in \Lambda\}$ and each $T_\lambda$ is a ring of fractions of $D$. Then the following are equivalent:*

(i) *$\boldsymbol{c}_D(fg)^\star = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star$ for all $0 \neq f, g \in K[X]$.*
(ii) *Each $T_\lambda \in \boldsymbol{\mathcal{T}}$ is a Prüfer domain.*
(iii) *$D$ is a P$\star$MD.*

Since the $w$–operation is the (semi)star operation on $D$ induced by the quotient rings $\boldsymbol{\mathcal{T}} := \{D_Q \mid Q \in \mathrm{Max}^t(D)\}$, i.e., $w = \wedge_{\boldsymbol{\mathcal{T}}}$, and since $w$ is of finite type, we have the following application of the previous corollary.

**Corollary 1.6.** *An integral domain $D$ is a P$v$MD if and only if $\boldsymbol{c}_D(fg)^w = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^w$ for all $0 \neq f, g \in K[X]$.*

Proof. Apply Corollary 1.5 and recall that, as a consequence of the fact that P$\widetilde{\star}$MD = P$\star$MD, we have P$w$MD = P$v$MD. □

This corollary on the one hand gives a nice general characterization of P$v$MD's, and on the other hand it establishes the "superiority" of the $w$–operation over the $t$–operation. As a matter of fact, since $F^t = F^v$ for each finitely generated nonzero ideal $F$, by [48, Lemme 1], we have:

$$\begin{aligned} D \text{ is integrally closed } &\Leftrightarrow \quad \boldsymbol{c}_D(fg)^v = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^v \text{ for all } 0 \neq f, g \in K[X] \\ &\Leftrightarrow \quad \boldsymbol{c}_D(fg)^t = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^t \text{ for all } 0 \neq f, g \in K[X]. \end{aligned}$$

In other words, for a Gaussian-like characterization of P$v$MD's, $w$ can do what $t$ cannot do.

As noted in the introduction, P$*$MD's were introduced by Houston, Malik, and Mott in [37] for a finite type star operation $*$. Note that for any star operation $*$, a $*$-invertible $*$-ideal is a $v$-ideal (cf. [39, Corollaire 1, page 21], [10, Proposition 3.1]). Now since in a P$*$MD every star ideal of finite type is $*_f$-invertible, and so is a $v$-ideal of finite type, we conclude that in a P$*$MD, where $*$ is a finite type star operation, every $*$-ideal is in fact a $t$-ideal.

It follows immediately by definition that for two semistar operations $\star_1$ and $\star_2$ on $D$, if $\star_1 \leq \star_2$ and if $D$ is a P$\star_1$MD, then $D$ is also a P$\star_2$MD. In particular, for each (semi)star operation $\star$ on $D$, we have that $D$ is a P$\star$MD implies that $D$ is also a P$v$MD since $\star \leq v$. Given a P$v$MD $D$, one wonders if there is a non-trivial (semi)star operation $\star$ of finite type on $D$ such that $D$ is not a P$\star$MD. Fontana, Jara, and Santos provided such an example in [24, Example 3.4]. The next corollary shows the way to construct more examples.

**Corollary 1.7.** *Let $D$ be a P$v$MD, let $n \geq 1$, and let $\boldsymbol{\mathcal{T}} := \{D_{S_i} \mid 1 \leq i \leq n\}$ be a finite family of quotient rings of $D$ such that $D = \bigcap_{i=1}^n D_{S_i}$. If some $D_{S_i}$ is not a Prüfer domain, then $D$ is a P$v$MD with a stable (semi)star operation $\star$ of finite type (e.g., $\star := \wedge_{\boldsymbol{\mathcal{T}}}$) such that $D$ is not a P$\star$MD.*

*Proof.* Let $\star := \wedge_{\boldsymbol{\mathcal{T}}}$ be the stable (semi)star operation induced by the family of overrings $\boldsymbol{\mathcal{T}}$, and suppose that $D_{S_1}$ is not a Prüfer domain. Then since $\boldsymbol{\mathcal{T}}$ is finite, and hence of finite character, $\star$ is a (semi)star operation of finite type [2, Theorem 2 (4)]. By Corollary 1.5 ((iii)$\Rightarrow$(ii)), $D$ is not a P$\star$MD. □

By applying Corollary 1.7, the next corollary provides further examples of P$v$MD's which are not P$\star$MD's for some stable (semi)star operation $\star$ of finite type. For the following statement, we fix a notation: given a $0 \neq x \in D$, we let $D_x$ be the quotient ring $D_S$, where $S := \{x^k \mid k \geq 0\}$.

**Corollary 1.8.** *Let $D$ be a P$v$MD. Then the following hold:*

(a) *Suppose that $D$ has nonzero nonunits $x_1, x_2, \ldots, x_n$ with $(x_1, x_2, \ldots, x_n)^v = D$, $n \geq 2$, and $D_{x_i}$ is not a Prüfer domain for some $i$. Then there is a stable (semi)star operation $\star$ of finite type on $D$ such that $D$ is not a P$\star$MD.*

(b) *Suppose that $M$ is a maximal ideal of $D$ with $D_M$ not a valuation domain. If there is a nonunit $x \in D \backslash M$, then $D$ has a stable (semi)star operation $\star$ of finite type such that $D$ is not a P$\star$MD.*

*Proof.* (a) The proof hinges on the fact that $(x_1, x_2, \ldots, x_n)^v = D$ if and only if $D = \bigcap_{i=1}^n D_{x_i}$ [56, Theorem 6]. Now the same procedure as in Corollary 1.7 does the rest of the job.

(b) Note that there is a $y \in M$ such that $(x, y) = D$. So, as in (a), we have $D = D_x \cap D_y$, and $D_x$ is not a Prüfer domain since $D_x \subseteq D_M$. □

Corollary 1.8(b) can be applied for instance to a non-quasilocal Krull domain of dimension two. In particular, take $D := K[X, Y]$, where $K$ is a field and $X, Y$ are two indeterminates over $K$. Clearly $D$ is a non-Prüfer P$v$MD. Let $M := (X +$

$1, Y)D$. Observe that $X \in D \setminus M$ is a nonunit, $D_M$ is a Noetherian regular local domain of dimension two (and thus it is not a valuation domain), and that, for instance, $(X, X+1)D = D$.

On the other hand, there do exist examples of non-Prüfer P$v$MD's $D$ such that for each pair of nonunits $x, y \in D$ with $((x,y)D)^v = D$, we have that $D_x$ and $D_y$ are both Prüfer domains. For instance, take a two-dimensional quasilocal Krull domain, e.g., $D := K[\![X,Y]\!]$, where $K$ is a field. (In this case, if $\alpha, \beta \in D$ are nonunits such that $((\alpha,\beta)D)^v = D$, then $D_\alpha$ and $D_\beta$ are Dedekind domains and $D = D_\alpha \cap D_\beta$.)

In the final part of this section, we examine the case of semistar operations of the type $\star = \wedge_{\boldsymbol{\mathcal{T}}}$ without assuming finite character.

**Proposition 1.9.** *Let $D$ be an integral domain with quotient field $K$ and let $\star$ be the semistar operation induced by a family $\boldsymbol{\mathcal{T}}$ of overrings of $D$, i.e., $\star = \wedge_{\boldsymbol{\mathcal{T}}}$. Consider the following statements:*

(i) $\boldsymbol{c}_D(fg)^\star = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star$ *for all* $0 \neq f, g \in K[X]$.

(ii) *Each overring $T \in \boldsymbol{\mathcal{T}}$ is a Prüfer domain.*

(iii) $(FF^{-1})^\star = D^\star$ *for all* $F \in \boldsymbol{f}(D)$.

*Then* (iii)⇒(ii)⇔(i). *Moreover, if we assume that each $T \in \boldsymbol{\mathcal{T}}$ is a quotient overring of $D$, then* (iii)⇔(ii)⇔(i).

*Proof.* Since $\star = \wedge_{\boldsymbol{\mathcal{T}}}$, it is easy to see that $\boldsymbol{c}_D(h)^\star T = \boldsymbol{c}_D(h)T = \boldsymbol{c}_T(h)$ for all $0 \neq h \in K[X]$ and all $T \in \boldsymbol{\mathcal{T}}$.

(i)⇒(ii) Apply the Gauss-Gilmer-Tsang Theorem to each $T \in \boldsymbol{\mathcal{T}}$.

(ii)⇒(i) Since we are assuming that each overring $T \in \boldsymbol{\mathcal{T}}$ is a Prüfer domain, we have $\boldsymbol{c}_D(fg)^\star = \bigcap\{\boldsymbol{c}_T(fg) \mid T \in \boldsymbol{\mathcal{T}}\} = \bigcap\{\boldsymbol{c}_T(f)\boldsymbol{c}_T(g) \mid T \in \boldsymbol{\mathcal{T}}\} = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star$ for all $0 \neq f, g \in K[X]$.

(iii)⇒(i) The proof is based on the Dedekind-Mertens Lemma, and it is analogous to the proof of Theorem 1.1 ((iii)⇒(i)).

Assume that each $T \in \boldsymbol{\mathcal{T}}$ is a quotient overring of $D$.

(ii)⇒(iii) Since each $T \in \boldsymbol{\mathcal{T}}$ is a Prüfer flat overring of $D$, every nonzero finitely generated fractional ideal of $T$ is invertible and $F^{-1}T = (FT)^{-1}$ for all $F \in \boldsymbol{f}(D)$. Therefore,

$$\begin{aligned}(FF^{-1})^\star = &\ \bigcap\{(FF^{-1})T \mid T \in \boldsymbol{\mathcal{T}}\} = \bigcap\{FT(FT)^{-1} \mid T \in \boldsymbol{\mathcal{T}}\} = \\ = &\ \bigcap\{T \mid T \in \boldsymbol{\mathcal{T}}\} = D^\star .\end{aligned}$$

□

**Corollary 1.10.** *Let $D$ be an integral domain with quotient field $K$. Set $E^b := \bigcap\{EV \mid V$ valuation overring of $D\}$ for all $E \in \boldsymbol{F}(D)$ (i.e., $E^b$ is the completion of the $D$–module $E$ in the sense of Zariski and Samuel* [60, Definition 1, page 347]*). If $D$ is integrally closed, then $\boldsymbol{c}_D(fg)^b = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^b$ for all $0 \neq f, g \in K[X]$.*

*Proof.* Let $\boldsymbol{\mathcal{T}}$ be the set of all valuation overrings of $D$. Clearly $b = \wedge_{\boldsymbol{\mathcal{T}}}$ and $b$ is a (semi)star operation on the integrally closed domain $D$ by Krull's Theorem [31, Theorem 19.8]. The statement then follows from Proposition 1.9 ((ii)⇒ (i)). □

Recall that an integral domain $D$ is called an *essential domain* if there exists a set of prime ideals $\Delta$ of $D$ such that $D = \bigcap\{D_P \mid P \in \Delta\}$ and $D_P$ is a valuation domain for each $P \in \Delta$. The set $\Delta$ is called a *set of essential prime ideals for* $D$. Every P$v$MD is essential, and an essential domain having a set of essential primes $\Delta$ of finite character (i.e, every nonzero element of $D$ is a nonunit in only finitely many $D_P$, $P \in \Delta$) is necessarily a P$v$MD [33, pages 717-718]. In [36] Heinzer and Ohm gave an example of an essential domain which is not a P$v$MD. For more examples of non-P$v$MD essential domains consult Zafrullah [57].

**Corollary 1.11.** *Let $D$ be an integral domain with quotient field $K$. Assume that $D$ is an essential domain and let $\Delta$ be a set of essential prime ideals for $D$. Then $\boldsymbol{c}_D(fg)^{\star_\Delta} = (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^{\star_\Delta}$ for all $0 \neq f, g \in K[X]$. However, if $D$ is neither a Prüfer domain nor a quasilocal domain, then there exists a stable (semi)star operation $\star$ of finite type on $D$, defined by a family $\boldsymbol{\mathcal{T}}$ of quotient overrings of $D$ (i.e., $\star := \wedge_{\boldsymbol{\mathcal{T}}}$), such that $\boldsymbol{c}_D(fg)^\star \neq (\boldsymbol{c}_D(f)\boldsymbol{c}_D(g))^\star$ for some $0 \neq f, g \in K[X]$. In particular, $D$ is not a P$\star$MD.*

*Proof.* The first statement follows from Proposition 1.9 ((ii)⇒(i)). Now assume that $D$ is not quasilocal. If $M$ is a maximal ideal of $D$, then we can find an element $x \in M$ such that $y := 1+x$ is not a unit in $D$. Therefore, we have found two nonzero nonunits $x, y \in D$ such that $(x, y) = D$, and thus $D = D_x \cap D_y$. If $D$ is not a Prüfer domain, then there exists a maximal ideal $N$ of $D$ such that $D_N$ is not a valuation domain. Since at least one of $x, y$ must avoid $N$ (i.e., $D_x \subset D_N$ or $D_y \subset D_N$), then $D_x$ or $D_y$ is not a Prüfer domain. Set $\boldsymbol{\mathcal{T}} := \{D_x, D_y\}$ and $\star := \wedge_{\boldsymbol{\mathcal{T}}}$. Clearly $\star$ is a stable (semi)star operation of finite type on $D$. The conclusion follows from Proposition 1.9 ((i)⇒(ii)) and Theorem 1.1 ((i)⇒(iii)). □

## 2. Class Groups

A somewhat interesting use of the results of Section 1 can be made, yet we need to introduce some terminology. While introducing the necessary terminology, we include some general facts that either link this work with the literature or illuminate some aspects of the theory of class groups. This, apparently discursive, treatment is also included to make a case for studying $\ast$–class groups for star operations $\ast$ different from $t$.

Let $\mathrm{Inv}^t(D)$ be the set of $t$–invertible $t$–ideals of an integral domain $D$. Clearly $\mathrm{Inv}^t(D)$ is an abelian group under $t$–multiplication and $\mathrm{Inv}^t(D)$ contains $\mathrm{Prin}(D)$, the set of nonzero principal fractional ideals of $D$, as a subgroup. The quotient-group $\mathrm{Cl}^t(D) := \mathrm{Inv}^t(D)/\mathrm{Prin}(D)$ is called the *$t$–class group of $D$* (note that it was introduced in [13] as "the class group" of the arbitrary domain $D$). The $t$–class group has the interesting property that while it is defined for any integral domain $D$, it is the divisor class group of $D$ when $D$ is a Krull domain and the ideal class group of $D$ when $D$ is a Prüfer domain. (Recall that in a Krull (resp., Prüfer) domain $D$, the nonzero fractional divisorial ideals $\boldsymbol{F}^v(D)$ (resp., nonzero finitely generated fractional ideals $\boldsymbol{f}(D)$) form an abelian group under the $v$–operation (resp., $d$-operation, i.e., usual product of ideals); the *divisor class group* (resp., *ideal class group*) *of* $D$ is the quotient-group $\boldsymbol{F}^v(D)/\mathrm{Prin}(D)$ (resp., $\boldsymbol{f}(D)/\mathrm{Prin}(D)$).

Moreover, a P$v$MD $D$ is a GCD-domain if and only if $\mathrm{Cl}^t(D)$ is trivial [14, Corollary 1.5]. There are other results that indicate that $\mathrm{Cl}^t(D)$ is intimately related with the divisibility properties of $D$, see e.g., [13], [14], [53], [8], and [11]. For these reasons, apparently, Halter-Koch [34] adapted the notion of the $t$–class group for monoids. In [10], D.F. Anderson surveyed the topic and introduced a generalization of $\mathrm{Cl}^t(D)$ by noting that if $*$ is a star operation on $D$, then the set $\mathrm{Inv}^*(D)$ of $*$–invertible $*$–ideals is an abelian group under $*$–multiplication and indeed $\mathrm{Prin}(D)$ is a subgroup of $\mathrm{Inv}^*(D)$. The quotient group $\mathrm{Cl}^*(D) := \mathrm{Inv}^*(D)/\mathrm{Prin}(D)$ is called the *$*$–class group of $D$.*

It is also possible to define a $\star$–class group for a semistar operation $\star$ on an integral domain $D$, but the generalization is not straightforward.

Let $\star$ be a semistar operation on $D$. We say that $I \in \overline{\boldsymbol{F}}(D)$ is *quasi–$\star$–invertible* (resp., *$\star$–invertible*) if $(I(D^\star : I))^\star = D^\star$ (resp., if $I \in \boldsymbol{F}(D)$ and $(I(D : I))^\star = D^\star$). It is obvious that $\star$–invertible implies quasi–$\star$–invertible, but the converse does not hold (even if $\star$ is a stable semistar operation of finite type) [28, Example 2.9]. However, it is clear from the definition that if $\star$ is a (semi)star operation and if $I \in \overline{\boldsymbol{F}}(D)$ is quasi–$\star$ –invertible, then $I$ must belong to $\boldsymbol{F}(D)$, and so $I$ is $\star$–invertible. It is not hard to prove that $I$ is quasi–$\star$ –invertible if and only if there exists an $H \in \overline{\boldsymbol{F}}(D)$ such that $(IH)^\star = D^\star$ [28, Lemma 2.10].

In the following proposition, we recall some known facts on $\star$ –invertibility and quasi–$\star$–invertibility (cf. [28, Propositions 2.15, 2.16, 2.18 and Corollary 2.17]).

**Proposition 2.1.** *Let $\star$ be a semistar operation on an integral domain $D$.*

(1) *Let $I \in \overline{\boldsymbol{F}}(D)$. Then $I$ is quasi–$\star_{_f}$–invertible if and only if $I$ and $(D^\star : I)$ are $\star_{_f}$–finite (hence, $\star$–finite) and $I$ is quasi–$\star$–invertible.*

*For the following statements, we assume $I \in \boldsymbol{F}(D)$.*

(2) *Let $I$ be quasi–$\star$–invertible. Then $I$ is $\star$–invertible if and only if $(D : I)^\star = (D^\star : I)$ (i.e., $\left(I^{-1}\right)^\star = (I^\star)^{-1}$).*
(3) *If $\star$ is a (semi)star operation, then $I$ is quasi–$\star$–invertible if and only if $I$ is $\star$–invertible.*
(4) *If $\star$ is stable and $I \in \boldsymbol{f}(D)$, then $I$ is quasi–$\star$–invertible if and only if $I$ is $\star$–invertible.*
(5) *$I$ is $\star_{_f}$–invertible if and only if $I$ is $\tilde{\star}$–invertible.*

If $\star$ is a semistar operation on an integral domain $D$, then we can introduce a semistar multiplication " $\times^\star$ " (or, simply, "$\times$" , if there is no danger for ambiguity) on the set $\mathrm{Inv}^\star(D) := \{I \in \overline{\boldsymbol{F}}(D) \mid I \text{ is } \star\text{–invertible and } I = I^\star\}$ by $I \times^\star J := (IJ)^\star$. Note that $(\mathrm{Inv}^\star(D), \times)$ is not a group in general, because, for instance, it does not have an identity element (e.g., when $D^\star \in \overline{\boldsymbol{F}}(D) \smallsetminus \boldsymbol{F}(D)$).

On the other hand, $\mathrm{QInv}^\star(D) := \{I \in \overline{\boldsymbol{F}}(D) \mid I \text{ is quasi–}\star\text{–invertible and } I = I^\star\}$, with the semistar multiplication "$\times$" introduced above, is always an abelian group with identity $D^\star$ and the unique inverse of $I \in \mathrm{QInv}^\star(D)$ is the $D$–module

$(D^\star : I) \in \overline{\boldsymbol{F}}(D)$ (it is not hard to prove that $(D^\star : I)$ belongs to $\mathrm{QInv}^\star(D)$). This fact also provides one of the motivations, in the semistar operation setting, for introducing and studying $\mathrm{QInv}^\star(D)$ (and not just $\mathrm{Inv}^\star(D)$, as in the "classical" star operation case). Moreover, it is not difficult to prove that $(\mathrm{Inv}^\star(D), \times)$ is a group if and only if $(D : D^\star) \neq (0)$ [28, page 657].

In particular, if $\star$ is a (semi)star operation on $D$, then as we have already observed, the notions of quasi–$\star$–invertible and $\star$–invertible coincide. More precisely, in this case, we have:

$$\mathrm{QInv}^\star(D) = \mathrm{Inv}^\star(D) = \{I \in \boldsymbol{F}(D) \mid I \text{ is } \star\text{–invertible and } I = I^\star\}\,.$$

Let $*$ be a star operation on an integral domain $D$. It is well known that every $*$–invertible $*$–ideal is a $v$–invertible $v$–ideal (see e.g., [59, Theorem 1.1(a)]). This property has a semistar analog. Given a semistar operation $\star$ on $D$, it is easy to see that the operation defined by

$$E^{v(\star)} := (D^\star : (D^\star : E)) \quad \text{for all } E \in \overline{\boldsymbol{F}}(D)$$

is a semistar operation on $D$ and $\star \leq v(\star)$ [47, Section 1.2.5]. Set $t(\star) := v(\star)_{_f}$. It is obvious that when $\star$ is a (semi)star operation, then $v(\star)$ (resp., $t(\star)$) coincides with the "classical " $v$–operation (resp., $t$ –operation) on $\boldsymbol{F}(D)$.

**Proposition 2.2.** [28, Corollary 2.12] *Let $\star$ be a semistar operation on an integral domain $D$ and let $I \in \overline{\boldsymbol{F}}(D)$. If $I$ is quasi–$\star$–invertible, then $I$ is quasi–$v(\star)$–invertible and $I^\star = I^{v(\star)}$. In particular,* $\mathrm{QInv}^\star(D)$ *is a subgroup of* $\mathrm{QInv}^{v(\star)}(D)$.

From the previous proposition (and its proof) and from Proposition 2.1(1), we easily deduce that if $I$ is quasi–$\star_{_f}$–invertible, then $I$ is quasi–$t(\star)$–invertible and $I^{\star_f} = I^{t(\star)}$ (cf. also [47, Corollary 3.13]). In particular, $\mathrm{QInv}^{\star_f}(D)$ is a subgroup of $\mathrm{QInv}^{t(\star)}(D)$.

At this point, it is clear that we can also define class groups in the semistar operation setting. We define the *$\star$–qclass group of $D$* to be the abelian group $\mathrm{QCl}^\star(D) := \mathrm{QInv}^\star(D)/\mathrm{Prin}(D)$, and under the assumption $(D : D^\star) \neq (0)$, we define the *$\star$–class group of $D$* to be the abelian group $\mathrm{Cl}^\star(D) := \mathrm{Inv}^\star(D)/\mathrm{Prin}(D)$. Clearly if $\star$ is a (semi)star operation, then $\mathrm{QCl}^\star(D) = \mathrm{Cl}^\star(D)$. When $\star = d$ is the identity (semi)star operation, then as in the classical case, we define the *Picard group of $D$* to be the abelian group $\mathrm{Pic}(D) := \mathrm{QCl}^d(D) = \mathrm{Cl}^d(D) := \mathrm{Inv}^d(D)/\mathrm{Prin}(D)$, where $\mathrm{Inv}^d(D)$ coincides with the group of the "usual" fractional invertible ideals of $D$. Note that $\mathrm{Inv}^d(D)$ is a subgroup of $\mathrm{QInv}^\star(D)$ for each semistar operation $\star$ on $D$ (resp., of $\mathrm{Inv}^\star(D)$ for each semistar operation $\star$ on $D$ such that $(D : D^\star) \neq (0)$). Therefore, following the classical case considered in [13], we call the quotient-group $\mathrm{QG}^\star(D) := \mathrm{QCl}^\star(D)/\mathrm{Pic}(D)$ (resp., $\mathrm{G}^\star(D) := \mathrm{Cl}^\star(D)/\mathrm{Pic}(D)$ for each semistar operation $\star$ on $D$ such that $(D : D^\star) \neq (0)$) the *local $\star$–qclass group* (resp., *local $\star$–class group*) *of $D$.* It is straightforward that when $\star = t$, the group $\mathrm{QG}^t(D) = \mathrm{G}^t(D)$ coincides with the "classical" local class group $\mathrm{G}(D)$ [13].

From Proposition 2.2, we deduce that for each semistar operation $\star$ on an integral domain $D$, we have $\mathrm{QCl}^\star(D) \subseteq \mathrm{QCl}^{v(\star)}(D)$, and under the assumption $(D :$

$D^\star) \neq (0)$, we have $\mathrm{Cl}^\star(D) \subseteq \mathrm{Cl}^{v(\star)}(D)$. When $\star$ is a semistar operation of finite type, then the previous inclusions can be replaced by $\mathrm{QCl}^\star(D) \subseteq \mathrm{QCl}^{t(\star)}(D)$ and $\mathrm{Cl}^\star(D) \subseteq \mathrm{Cl}^{t(\star)}(D)$, respectively. Furthermore, if $\star$ is a (semi)star operation (resp., a (semi)star operation of finite type), then we have $\mathrm{Pic}(D) \subseteq \mathrm{Cl}^\star(D) \subseteq \mathrm{Cl}^v(D)$ (resp., $\mathrm{Pic}(D) \subseteq \mathrm{Cl}^\star(D) \subseteq \mathrm{Cl}^t(D)$).

For the purposes of the present section, from now on we will only consider the "classical" case of a star operation $*$. If $*$ is a star operation of finite type on $D$, then $\mathrm{Cl}^*(D)$ provides an interesting generalization of the ($t$–)class group, but

(a) $t$–invertibility being somewhat abundant and more closely linked with divisibility [59], the $t$–class group seems to have more applications, especially in view of its similarity to the divisor class group for Krull domains and the facility of the $t$–operation with polynomial rings and with rings of fractions.

On the other hand,

(b) there are very few examples of $*$–class groups that are not $d$–class groups, $t$–class groups, or $v$–class groups.

Of course we cannot do much about (a), but we can use Corollary 1.8 to give examples of $*$–class groups such that $\mathrm{Cl}^*(D) \subsetneq \mathrm{Cl}^t(D)$ and of $\mathrm{Cl}^*(D) = \mathrm{Cl}^t(D)$, where there is at least one nonzero finitely generated ideal $F$ of $D$ such that $F$ is $t$–invertible, but not $*$–invertible.

**Proposition 2.3.** *Let $D$ be an integral domain in which every $t$–invertible $t$–ideal is invertible. Then* $\mathrm{Pic}(D) = \mathrm{Cl}^*(D) = \mathrm{Cl}^t(D)$ *for any star operation $*$ of finite type on $D$.*

*Proof.* Indeed, we have already observed that every $*$–invertible $*$–ideal is a $t$–invertible $t$–ideal and every invertible ideal is a $*$–invertible $*$–ideal for any star operation $*$ on $D$. So $\mathrm{Inv}(D) \subseteq \mathrm{Inv}^*(D) \subseteq \mathrm{Inv}^t(D)$. On the other hand, every $t$–invertible $t$–ideal is invertible by hypothesis. Combining these inclusions, we conclude that $\mathrm{Inv}(D) = \mathrm{Inv}^*(D) = \mathrm{Inv}^t(D)$. Hence $\mathrm{Pic}(D) = \mathrm{Cl}^*(D) = \mathrm{Cl}^t(D)$ in this case. □

Now are there any integral domains that satisfy the hypothesis of Proposition 2.3 ? Indeed, there are plenty. Recall from [55] that an integral domain $D$ is *a pre-Schreier domain* if for $0 \neq x, y, z \in D$, $x|yz$ implies that $x = rs$ for some $r, s \in D$ with $r|y$ and $s|z$. Pre-Schreier domains are a generalization of GCD-domains (cf. [16] and [18, Theorem 1]). It was indicated in [14, Proposition 1.4] that if $D$ is a pre-Schreier domain, then $\mathrm{Cl}^t(D) = 0$. Obviously if $D$ is a pre-Schreier domain and $*$ is a star operation of finite type on $D$, then we must have $\mathrm{Cl}^*(D) = \mathrm{Cl}^t(D) = 0$. We are aiming at a somewhat more general result:

**Corollary 2.4.** *Let $D$ be an integral domain. If* $\mathrm{Cl}^t(D_M) = 0$ *for all maximal ideals $M$ of $D$ (e.g., if $D$ is a locally GCD-domain* [14]*), then* $\mathrm{Pic}(D) = \mathrm{Cl}^*(D) = \mathrm{Cl}^t(D)$ *for any star operation $*$ of finite type on $D$.*

*Proof.* Indeed, the local class group $\mathrm{G}(D) = \mathrm{Cl}^t(D)/\mathrm{Pic}(D) = 0$ if $\mathrm{G}(D_M) = 0$ (in particular, if $\mathrm{Cl}^t(D_M) = 0$) for each maximal ideal $M$ of $D$ [14, Proposition 2.4]. □

This leaves us with the task of providing an example of an integral domain $D$ such that $\mathrm{Cl}^*(D) \subsetneq \mathrm{Cl}^t(D)$ (also see Example 3.10). For this, we recall that an integral domain $D$ is a *generalized GCD* (for short, *G-GCD*) *domain* if for every nonzero finitely generated ideal $F$ of $D$, we have that $F^v$ is invertible [5]. Moreover, $D$ is a G-GCD domain if and only if $D$ is a P$v$MD which is a locally GCD domain [55, Corollary 3.4]. So if we are looking for a P$v$MD example of an integral domain $D$ with $\mathrm{Cl}^*(D) \subsetneq \mathrm{Cl}^t(D)$, then $D$ had better not be a G-GCD domain.

**Proposition 2.5.** *Let $D$ be a P$v$MD such that there are nonzero nonunits $x_1, x_2, \dots, x_n \in D$ with $((x_1, x_2, \dots, x_n)D)^v = D$. Suppose that for at least one $j$, $1 \le j \le n$, $D_{x_j}$ is not a G-GCD domain. Let $*$ be the stable star operation of finite type induced on $D$ by the finite family of overrings $\boldsymbol{\mathcal{T}} := \{D_{x_i} \mid 1 \le i \le n\}$, i.e., $* = \wedge_{\boldsymbol{\mathcal{T}}}$, or equivalently, $I^* := \bigcap_{i=1}^n ID_{x_i}$ for each $I \in \boldsymbol{F}(D)$. Then there exists a nonzero finitely generated ideal $F$ of $D$ such that $F^v$ is not a $*$–invertible ($*$–)ideal. Consequently, in $D$ we have $\mathrm{Cl}^*(D) \subsetneq \mathrm{Cl}^t(D)$.*

*Proof.* We note that given a nonzero finitely generated ideal $F$ of $D$, for $F^v$ to be $*$–invertible it is essential that $(F^{-1}F^v)D_{x_i} = D_{x_i}$ for each $i = 1, 2, \dots, n$. Since for say $i = j$, $D_{x_j}$ is not a G-GCD domain, we conclude that there is a finitely generated ideal $H$ of $D_{x_j}$ such that $H^{v_j}$ is not invertible in $D_{x_j}$ (where $v_j$ denotes the $v$–operation on $D_{x_j}$). But as $H$ is finitely generated in $D_{x_j}$, we can assume that $H = FD_{x_j}$, where $F$ is a nonzero finitely generated ideal of $D$. But, as $D$ is a P$v$MD, $F$ is a $t$–invertible ideal of $D$; thus we have $H^{v_j} = F^vD_{x_j}$ [14]. Since $H^{v_j}$ is not invertible, we easily conclude that $F^{-1}F^vD_{x_j} \neq D_{x_j}$. Hence $(F^{-1}F^v)^* \neq D$. The "consequently" part is obvious. □

It seems important to also indicate the situations where there are two distinct star operations, made from a general star operation $*$, say $\mu(*)$ and $\nu(*)$, where $\mu(*) \neq \nu(*)$, but $\mathrm{Cl}^{\mu(*)}(D) = \mathrm{Cl}^{\nu(*)}(D)$. The constructions that we have in mind are the $*_f$ and the $\widetilde{*}$ $(= *_w)$ constructions from a general star operation $*$ mentioned in the introduction. Now $*_f$ and $\widetilde{*}$ are not always equal, but as a consequence of Proposition 2.1(5), $\mathrm{Inv}^{\widetilde{*}}(D) = \mathrm{Inv}^{*_f}(D)$. Thus:

**Proposition 2.6.** *Let $*$ be a star operation on an integral domain $D$. Then $\mathrm{Cl}^{\widetilde{*}}(D) = \mathrm{Cl}^{*_f}(D)$. In particular, $\mathrm{Cl}^w(D) = \mathrm{Cl}^t(D)$.* □

Let $\boldsymbol{F}^v(D) := \{I \in \boldsymbol{F}(D) \mid I = I^v\}$ be the set of divisorial fractional ideals of $D$. It is well known that $\boldsymbol{F}^v(D)$ is a group under the $v$–multiplication, $\times^v$, if and only if $D$ is completely integrally closed [31, Theorem 34.3]. In this situation, the group $\boldsymbol{F}^v(D)/\mathrm{Prin}(D)$ is called the *divisor class group of $D$.* The $t$–class group has often been dubbed as a generalization of the divisor class group because, as we remarked above, $\mathrm{Cl}^t(D)$ is precisely the divisor class group for a Krull domain $D$. But the $t$–class group is in general far away from the divisor class group (when defined). For instance, for a completely integrally closed domain $D$, the divisor class group of $D$ is zero only if $I^v$ is principal for every $I \in \boldsymbol{F}(D)$. However, there do exist completely integrally closed GCD-domains (in fact, rank-one valuation domains) which contain nonprincipal proper $v$–ideals. One example that comes to mind is a rank-one valuation domain $V$ with value group $\mathbb{Q}$ the rationals. In this case, the divisor class group of $V$ is nonzero (see [54, Example 2.7] for an elementary verification), but as $V$ is a GCD-domain, $\mathrm{Cl}^t(V)$ is zero [14, Example 1.2].

We next give a complete verification of the valuation domain example mentioned above. Let us start by noting that, for a valuation domain $D$ which is not a field, there are at most two distinct star operations on $D$, the $v$–operation and the operation $d = t$ [31, Exercise 12, p. 431]. Therefore, for a valuation domain $D$, we have $\mathrm{Cl}^t(D) = \mathrm{Pic}(D) = \{0\}$. Let us also note that a valuation domain $D$ is completely integrally closed if and only if $\dim(D) \leq 1$ [31, Theorem 17.5(3)], and in this case, as observed above, $\boldsymbol{F}^v(D)$ is a group under $v$–multiplication. Thus in this case, $\mathrm{Cl}^v(D)$ coincides with the divisor class group of $D$. Next, let $G$ be the value group of the valuation domain $D$ and let $\omega : K^\bullet \to G$ be the valuation that gives rise to $D$ (where $K^\bullet := K \setminus \{0\}$). When $\dim(D) = 1$, either $D$ is a DVR or $G$ is a dense subspace of the real numbers $\mathbb{R}$ (cf. [31, page 193] or [12, Chapitre 6, §4, N. 5, Propositions 7 et 8]).

**Theorem 2.7.** *Let $D$ be a (one-dimensional) valuation domain with value group $G \subseteq \mathbb{R}$. Then*

(1) *If $D$ is a DVR, then* $\mathrm{Pic}(D) = \mathrm{Cl}^v(D) = 0$.
(2) *If $D$ is not a DVR, then* $\mathrm{Pic}(D) = 0$ *and* $\mathrm{Cl}^v(D) = \mathbb{R}/G$.

*Proof.* Suppose that $D$ is not a DVR; so $G$ is dense in $\mathbb{R}$. Define a map $\varphi : \boldsymbol{F}(D) \to \mathbb{R}$ by $\varphi(I) := \sup\{\omega(x) \mid I \subseteq xD \text{ for } x \in K^\bullet\}$. Note that $\varphi$ is well-defined since

(a) $yD \subseteq I \subseteq xD$ for $x, y \in K^\bullet$ implies $\omega(x) \leq \omega(y)$,
(b) $G \subseteq \mathbb{R}$, with $\mathbb{R}$ complete, and
(c) $\varphi(xD) = \omega(x)$ for $x \in K^\bullet$.

Using these observations, we also note that $\varphi(I) = \sup\{\omega(x) \mid x \in K^\bullet \text{ and } \omega(x) \leq \omega(i) \text{ for all } 0 \neq i \in I\}$. Therefore, for all $0 \neq i \in I$, we have $\omega(i) \geq \varphi(I)$. For the same reasons, if $I \subseteq xD$, then $\omega(x) \leq \varphi(I)$. The proof of Theorem 2.7 then follows from the following four lemmas.

**Lemma 2.8.** *Let $D$ be as in Theorem 2.7. Then the following statements are equivalent for $I, J \in \boldsymbol{F}(D)$.*

(i) $\varphi(I) = \varphi(J)$.
(ii) $\{xD \mid I \subseteq xD \text{ for } x \in K^\bullet\} = \{xD \mid J \subseteq xD \text{ for } x \in K^\bullet\}$.
(iii) $I^v = J^v$.

*Proof.* Clearly (ii)⇔(iii) and (ii)⇒(i). For (i)⇒(ii), suppose that $\varphi(I) = \varphi(J)$, but there is an $x \in K^\bullet$ such that $I \subseteq xD$ and $J \nsubseteq xD$. Then $xD \subsetneq J$. So there is a $y \in J$ such that $I \subseteq xD \subsetneq yD \subseteq J$. But then as already noted, we have $\varphi(J) \leq \omega(y) < \omega(x) \leq \varphi(I)$, a contradiction. □

From the above lemma, it follows that $\varphi$ restricts to a (well-defined) injective map $\varphi : \boldsymbol{F}^v(D) \to \mathbb{R}$.

**Lemma 2.9.** *Let $D$ and $G$ be as in Theorem 2.7, and let $\varphi : \boldsymbol{F}^v(D) \to \mathbb{R}$ be defined as above. Then*
$$\varphi^{-1}(G) = \mathrm{Prin}(D)\,.$$

*Proof.* Let $I \in \varphi^{-1}(G)$. Then $\varphi(I) = \alpha \in G$; so there is an $x \in K^\bullet$ such that $\omega(x) = \alpha$. Since $\omega(x) = \varphi(xD)$, we have $\varphi(I) = \varphi(xD)$. By Lemma 2.8, we have $I^v = xD$. But as $I \in \boldsymbol{F}^v(D)$, we have $I = I^v$. Conversely, suppose that $xD \in \mathrm{Prin}(D)$. Then $\varphi(xD) = \omega(x) \in G$. □

**Lemma 2.10.** *Let $D$ and $G$ be as in Theorem 2.7. Then the map $\varphi : \boldsymbol{F}^v(D) \to \mathbb{R}$ defined above is surjective.*

*Proof.* Let $\alpha \in \mathbb{R}$. Define $I_\alpha := \bigcap\{xD \mid x \in K^\bullet \text{ with } \omega(x) \leq \alpha\}$. Since $G$ is dense in $\mathbb{R}$, there is a $y \in K^\bullet$ such that $\omega(y) > \alpha$. Therefore $yD \subseteq xD$ for each $x \in K^\bullet$ with $\omega(x) \leq \alpha$. This ensures, in particular, that $I_\alpha$ is nonzero, and so $I_\alpha \in \boldsymbol{F}^v(D)$. In order to establish the surjectivity, we show that $\varphi(I_\alpha) = \alpha$. Clearly $\varphi(I_\alpha) \geq \alpha$ because $G$ is dense in $\mathbb{R}$. Suppose that $\varphi(I_\alpha) = \beta > \alpha$. So there is a $z \in K^\bullet$ such that $I_\alpha \subseteq zD$ and $\alpha < \omega(z) = \gamma \leq \beta$. Let $\gamma' \in G$ such that $\alpha < \gamma' < \gamma \leq \beta$ and let $\omega(z') = \gamma'$ for some $z' \in K^\bullet$. Then for any $x \in K^\bullet$ with $\omega(x) \leq \alpha$, we have $\omega(x) < \omega(z')$, forcing $z' \in I_\alpha$. Now from $\gamma' < \gamma$, we have $\omega(z') < \omega(z)$, which is equivalent to $zD \subsetneq z'D$ $(\subseteq I_\alpha)$, a contradiction. □

**Lemma 2.11.** *The map $\varphi : \boldsymbol{F}^v(D) \to \mathbb{R}$ is a group homomorphism.*

*Proof.* Let $I, J \in \boldsymbol{F}^v(D)$. We show that $\varphi((IJ)^v) = \varphi(I) + \varphi(J)$. Let $I \subseteq xD$ and $J \subseteq yD$ for $x, y \in K^\bullet$. Then $IJ \subseteq xyD$, and hence $\gamma := \varphi((IJ)^v) = \varphi(IJ) \geq \omega(xy) = \omega(x) + \omega(y)$. Thus $\varphi((IJ)^v) \geq \varphi(I) + \varphi(J)$. Suppose that $\varphi(I) + \varphi(J) < \varphi((IJ)^v)$. Then there are $\alpha, \beta \in G$ such that $\varphi(I) \leq \alpha$, $\varphi(J) \leq \beta$, and $\alpha + \beta < \gamma$. Choose $x, y \in K^\bullet$ with $\omega(x) = \alpha$ and $\omega(y) = \beta$. Then $xD \subseteq I$ and $yD \subseteq J$; so $xyD \subseteq IJ$. Let $IJ \subseteq zD$ for $z \in K^\bullet$. Then as $xyD \subseteq zD$, we have $\omega(z) \leq \omega(xy) = \omega(x) + \omega(y) = \alpha + \beta < \gamma$. So $\gamma = \varphi((IJ)^v) = \sup\{\omega(z) \mid IJ \subseteq zD \text{ and } z \in K^\bullet\} \leq \omega(xy) = \alpha + \beta < \gamma$, a contradiction. □

Given a rank-one valuation domain $D$, if we assume in Theorem 2.7 that $D$ is not a DVR and that $G \neq \mathbb{R}$, then $\mathrm{Cl}^v(D)$ (which coincides in this case with the divisor class group of $D$) is not zero, whereas the $t$–class group $\mathrm{Cl}^t(D)$ $(= \mathrm{Pic}(D))$ is zero. Having shown that both the divisor class group and $t$–class group can coexist without being equal, we conclude that the $t$–class group is not a generalization of the divisor class group.

## 3. $v$–class groups and valuation domains

In the previous section, we have seen the divisor class group as the $v$–class group in the case of one-dimensional valuation domains (Theorem 2.7) and, more generally, for completely integrally closed domains [31, Theorems 17.5 (3) and 34.3]. But thanks to the generality of its definition, the group $\mathrm{Cl}^v(D)$ does not need $D$ to have any special properties. In other words, the $v$–class group is defined for any integral domain. Now let us note that the $v$–operation being the coarsest star operation, the $v$–class group (for $v \neq t$) has hitherto been neglected. So, we do not have a lot of examples from the literature to offer. However, to show that the $v$–class group has a life of its own and some interesting properties, we study the $v$–class group of some integral domains of interest.

Let us first start with some relevant cases where the $v$–class group is the same as the $t$–class group.

**Proposition 3.1.** (1) *Let $D$ be an integral domain such that $F^{-1}$ is of finite type for all $F \in \boldsymbol{f}(D)$. Then $\mathrm{Cl}^t(D) = \mathrm{Cl}^v(D)$.*

(2) *Let $D$ be a valuation domain (in particular, $0 = \mathrm{Pic}(D) = \mathrm{Cl}^t(D)$).*

- (2a) *Assume that $D$ is one-dimensional. Then $\mathrm{Cl}^v(D) = 0$ if and only if either $D$ is a DVR or $D$ has value group $\mathbb{R}$. Moreover, $\mathrm{Cl}^v(D)$ is a divisible abelian group and may have torsion elements. However, $\mathrm{Cl}^v(D)$ is torsion-free if the value group of $D$ is $\mathbb{Q}$.*
- (2b) *Assume that $D$ is an $n$-dimensional valuation domain, $1 \leq n \leq \infty$, with maximal ideal $M$. If $M$ is principal, then all nonzero fractional ideals of $D$ are divisorial. So $d = v$, and thus $0 = \mathrm{Pic}(D) = \mathrm{Cl}^t(D) = \mathrm{Cl}^v(D)$.*

*Proof.* The proof of (1) is straightforward since by Proposition 2.1 we deduce that $I \in \boldsymbol{F}(D)$ is $t$–invertible if and only if $I$ and $I^{-1}$ are $t$–finite and $I$ is $v$–invertible. The main examples of domains for which this result holds are Mori domains, which include Noetherian and Krull domains as special cases.

For the proof of (2a) consult Theorem 2.7. The other statements are easy consequences of the first one since $\mathbb{R}$ is an additive divisible group and any quotient group of a divisible group is again divisible. It is easy to check that $\mathbb{R}/\mathbb{Q}$ is torsion-free.

(2b) is well known [31, Example 12, page 431]. □

If $D$ is an $n$-dimensional valuation domain, $1 \leq n \leq \infty$, with maximal ideal $M$ and $M$ is not principal, we get a completely different story. We essentially devote the major part of the remainder of this section to this case. To give a clear idea of this situation, we start with an example.

**Example 3.2.** *Let $V$ be a nondiscrete rank-one valuation domain with value group $G$, let $K$ be the quotient field of $V$, let $X$ be an indeterminate over $K$, and let $D := V + XK[\![X]\!]$. Then $D$ is a two-dimensional valuation domain (with quotient field $K(\!(X)\!)$); thus $0 = \mathrm{Pic}(D) = \mathrm{Cl}^t(D)$, and moreover, $\mathrm{Cl}^v(D) = \mathbb{R}/G$.*

The fact that $D$ is a two-dimensional valuation domain follows from [31, Proposition 18.2(3)]. By using a very general theory of the class groups on pullback constructions [27], we have $\mathrm{Cl}^v(D) \cong \mathrm{Cl}^v(V)$ (see also the following Theorem 3.5). As a matter of fact [27, Corollary 2.7] ensures that, given a quasilocal integral domain $(T, M, k)$ and a proper subring $S$ of $k$, if $R$ is the integral domain arising from the following pullback of canonical homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ T & \longrightarrow & T/M = k\,, \end{array}$$

then $\mathrm{Cl}^v(S) \cong \mathrm{Cl}^v(R)$. The conclusion then follows from Theorem 2.7(2).

Since the case of valuation domains is rather peculiar and relevant, it deserves particular attention. In this case, in fact, it is possible to give direct proofs of special cases of more general results on $\ast$–class groups concerning pullback constructions (cf., in particular, [21] and [27]) by using elementary direct methods that are elegant and simple to handle. The next goal is to show how, in the previous Example 3.2, it is possible to avoid the use of [27, Corollary 2.7].

Recall that for each fractional ideal $I$ of an integral domain $D$ with quotient field $K$, we have $I^v = \bigcap\{zD \mid z \in K \text{ and } I \subseteq zD\}$. Moreover, as observed in [59, page 432], $I^v \neq D$ if and only if there are $0 \neq a, b \in D$ such that $I \subseteq (aD : bD)$ and $a \nmid b$. Also, recall from [9] that an integral domain $D$ is an *IP–domain* if every proper integral $v$–ideal of $D$ is an intersection of integral principal ideals of $D$.

Now let $I$ be a proper integral ideal of a valuation domain $V$ with quotient field $K$. Then $I^v = \bigcap\{zV \mid z \in K \text{ and } I \subseteq zV\}$. Since $V$ is a valuation domain and $I$ an integral ideal of $V$, we have $I^v \subseteq V$, and so

$$\begin{aligned} I^v = &\ (\textstyle\bigcap\{zV \mid z \in V \text{ and } I \subseteq zV\}) \cap (\textstyle\bigcap\{zV \mid z \in K \setminus V \text{ and } I \subseteq zV\}) \\ = &\ \textstyle\bigcap\{zV \mid z \in V \text{ and } I \subseteq zV\} \end{aligned}$$

(note that for $z \in K$, $z \in K \setminus V$ is equivalent to $z^{-1} \in V$, and hence $\bigcap\{zV \mid z \in K \setminus V \text{ and } I \subseteq zV\} = V$). So a valuation domain $V$ is an IP–domain. Indeed, if $I$ is nonzero principal, then $I = xV = I^v$. On the other hand, if $I$ is not principal, then $I$ is not finitely generated. This leads to two cases:

$$\text{(a) } I^v = V \qquad \text{or} \qquad \text{(b) } I^v = \textstyle\bigcap\{zV \mid z \in V \text{ and } I \subseteq zV \subsetneq V\}.$$

We now prepare to use the fact that if $V$ is a valuation domain and $P$ is a nonmaximal prime ideal of $V$, then $V/P$ is a valuation domain that is not a field.

**Lemma 3.3.** *Let $V$ be a valuation domain with maximal ideal $M$, $P \subsetneq M$ a prime ideal of $V$, and $I$ an integral ideal of $V$ with $P \subsetneq I$. Then $I^v/P = (I/P)^v$. In particular, $I/P$ is a $v$–ideal of $V/P$ if and only if $I$ is a $v$–ideal of $V$.*

*Proof.* For $z \in V$, we have $I \subseteq zV$ if and only if $I/P \subseteq (zV)/P = (zV+P)/P$. Assume that $P \subsetneq I \subseteq I^v \subsetneq V$. As above, $I^v/P = (\bigcap\{zV \mid z \in V \text{ and } I \subseteq zV \subseteq V\})\,/P$ $= \bigcap\{(zV)/P \mid z \in V \text{ and } I \subseteq zV \subseteq V\} = (I/P)^v$. □

To prove the main theorem of this section, we need to prove yet another lemma.

**Lemma 3.4.** *Let $V$ be a valuation domain with maximal ideal $M$, $P \subsetneq M$ a prime ideal of $V$, and $I$ an integral ideal of $V$ with $P \subsetneq I$. Then $I/P$ is a $v$–invertible $v$–ideal of $V/P$ if and only if $I$ is a $v$–invertible $v$–ideal of $V$.*

*Proof.* Let $I$ be a $v$–invertible ideal of $V$. Since $(II^{-1})^v = V$, we claim that $M \subseteq II^{-1} \subseteq V$. Of these, $II^{-1} \subseteq V$ always holds; so we concentrate on $M \subseteq II^{-1}$. If $I$ is principal, then $II^{-1} = V$, and so trivially $M \subseteq II^{-1}$. If on the other hand $I$ is not finitely generated, then $II^{-1} = Q$ is a prime ideal of $V$ [1, Theorem 1]. If $Q$ were nonmaximal, then $Q$ must be divisorial (with $Q = (V : V_Q)$), and so $V = (II^{-1})^v = Q \subsetneq V$, a contradiction. Thus $M = II^{-1}$, and in this case too, $M \subseteq II^{-1}$.

Next, since $P \subsetneq I$, there exists an element $j \in I\backslash P$. Let $J := jI^{-1}$; clearly $J \subseteq V$. We claim that $P \subsetneq J$. For if not, then as we are working in a valuation domain $V$, we must have $J = jI^{-1} \subseteq P$. Multiplying both sides by $I$ and applying the $v$–operation, we get $jV = j(II^{-1})^v = (jII^{-1})^v \subseteq P$, because $P$ is a $v$-ideal (being nonmaximal). This gives $j \in P$ a contradiction. Hence $P \subsetneq J$.

Now $(IJ)^v = jV$, and $I, J$, and $jV$ all properly contain $P$. So by Lemma 3.3, $(jV + P)/P = (jV)/P = (IJ)^v/P = (IJ/P)^v = ((I/P)(J/P))^v$, and this establishes that $I/P$ is $v$-invertible in $V/P$. Moreover, with an appeal again to Lemma 3.3, we conclude that if $I$ is a $v$–invertible $v$–ideal of $V$, then $I/P$ is a $v$–invertible $v$–ideal of $V/P$.

Conversely, if $I/P$ is a $v$–invertible $v$–ideal of $V/P$, then for some ideal $J$ of $V$ with $P \subsetneq J$ we have $((I/P)(J/P))^v = (xV)/P$, where $x \in V\backslash P$. From this fact, it is easy to deduce that $(IJ)^v = xV$, and so, $I$ is a $v$–invertible $v$–ideal of $V$. □

**Theorem 3.5.** *Let $V$ be a valuation domain with maximal ideal $M$ and proper quotient field $K$, and let $P \subsetneq M$ be a prime ideal of $V$. Then* $\mathrm{Cl}^v(V) \cong \mathrm{Cl}^v(V/P)$.

*Proof.* We first show that if there is a $v$–invertible $v$–ideal $I$ of $V$, then its class $[I]$ contains an integral ideal $J$ that properly contains $P$. Let $\mathfrak{J} \in \mathrm{Cl}^v(V)$. Then $\mathfrak{J} = [I]$ for some $v$–invertible $v$–ideal $I$ of $V$. Since $(II^{-1})^v = V$, as in the proof of Lemma 3.4, we have $P \subsetneq II^{-1}$. Thus $P \subsetneq jI$ for some $j \in I^{-1}$. Let $J := jI$. Define $\varphi(\mathfrak{J}) := [J/P]$. Note that $[J/P] \in \mathrm{Cl}^v(J/P)$ by Lemmas 3.3 and 3.4.

Since $[J] = [I]$ in $\mathrm{Cl}^v(V)$, it is enough to study the case of integral $v$–invertible $v$–ideals $I \supsetneq P$, in which case $\varphi([I]) = [I/P]$. We first show that $\varphi$ is well-defined. Let $A$ and $B$ be two $v$–invertible $v$–ideals of $V$ such that $P \subsetneq A, B \subseteq V$ and $[A] = [B]$. Then $A = tB$ for some $0 \neq t \in K$. Since $t \in V$ or $t^{-1} \in V$, we can assume that $t \in V$ (interchanging eventually $A$ with $B$). Once we assume that $t \in V$, we find that $t \in V\backslash P$ because $P \subsetneq A$. Thus $A/P = (tB)/P = ((t + P)/P)(B/P)$ in $V/P$. So, $[A/P] = [B/P]$ in $\mathrm{Cl}^v(V/P)$. Thus $\varphi$ is well-defined. Next we show that $\varphi$ is injective. Suppose that $[A/P] = [B/P]$ in $\mathrm{Cl}^v(V/P)$, where $P \subsetneq A, B \subseteq V$ are $v$–invertible $v$–ideals of $V$. Then as before we can assume that $A/P = ((t + P)/P)(B/P)$ for some $t \in V\backslash P$. Thus $A/P = (tB)/P$, which forces $A = tB$, and hence $[A] = [B]$ in $\mathrm{Cl}^v(V)$. To show that $\varphi$ is surjective, let $\mathfrak{J} \in \mathrm{Cl}^v(V/P)$. Then $\mathfrak{J} = [J]$ for some $v$–invertible integral $v$–ideal $J$ of $V/P$. By the above comments and Lemmas 3.3 and 3.4, $J = I/P$ for some $v$–invertible integral $v$–ideal $I$ of $V$ such that $P \subsetneq I$. Thus $\varphi([I]) = [I/P] = [J] = \mathfrak{J}$; so $\varphi$ is surjective. Finally, we show that $\varphi$ is a group-homomorphism. Let $[I], [I'] \in \mathrm{Cl}^v(V)$, where $P \subsetneq I, I' \subseteq V$ are $v$–invertible $v$–ideals of $V$. Then $\varphi([I] \cdot [I']) = \varphi([(II')^v]) = [(II')^v/P] = [(II'/P)^v]$

$= [((I/P)(I'/P))^v] = [I/P] \cdot [I'/P] = \varphi([I]) \cdot \varphi([I'])$. (Here we have used the fact that $P \subsetneq I, I'$ implies that $P \subsetneq II'$.) Thus $\varphi$ is an isomorphism. □

The next statement, which has already appeared in Proposition 3.1, can be easily reobtained as a consequence of Theorem 3.5.

**Corollary 3.6.** *Let $V$ be a valuation domain with principal maximal ideal $M$. Then* $\mathrm{Cl}^v(V) = 0$.

*Proof.* Let $M = qV$ and set $P := \bigcap_{n\geq 1} q^n V$. Then $P$ is a prime ideal of $V$, $P \subsetneq M$, and there is no prime ideal between $P$ and $M$ [31, Theorems 17.1 and 17.3]. This makes $V/P$ a discrete rank-one valuation domain, and so $\mathrm{Cl}^v(V/P) = 0$. But then, by Theorem 3.5, we have $\mathrm{Cl}^v(V) \cong \mathrm{Cl}^v(V/P) = 0$. □

Let $V$ be a valuation domain such that the maximal ideal $M$ of $V$ is *idempotent* (i.e., $M^2 = M$) and *branched* (i.e., has an $M$ primary ideal different from $M$). Recall that $M$ is idempotent if and only if $M$ is not finitely generated (i.e., not principal) and that $M$ is branched if and only if there is a prime ideal $P \subsetneq M$ such that there is no prime ideal between $P$ and $M$ [31, Theorem 17.3]. Let us call $P$ *the prime ideal directly below $M$*. Indeed, this makes $V/P$ a rank-one valuation domain. If $M$ is idempotent, then it is easy to check that $M^v = V$ [23, Corollary 3.1.3].

**Corollary 3.7.** *Let $V$ be a valuation domain with maximal ideal $M$ that is branched and idempotent, let $P$ be the prime ideal directly below $M$, and let $G$ be the value group of $V/P$. Then* $\mathrm{Cl}^v(V) \cong \mathbb{R}/G$.

*Proof.* By Theorem 3.5, we have $\mathrm{Cl}^v(V) \cong \mathrm{Cl}^v(V/P)$. Since $V/P$ is a nondiscrete rank-one valuation domain, its value group $G$ is isomorphic to a subgroup of $\mathbb{R}$, and thus we have $\mathrm{Cl}^v(V/P) = \mathbb{R}/G$ (Theorem 2.7 (2)). □

Note that Corollaries 3.6 and 3.7 let us compute $\mathrm{Cl}^v(V)$ for any finite-dimensional valuation domain $V$. Theorem 3.5 can also be used to give interesting statements relative to the $D + M$ construction of Gilmer. For instance, let $V'$ be a valuation domain that is expressible as $K + M'$, where $K$ is a field and $M'$ the maximal ideal of $V'$. Also let $V$ be a valuation domain with quotient field $K$. Then $D := V + M'$ is a valuation domain such that $\mathrm{Cl}^v(D) \cong \mathrm{Cl}^v(V)$.

The reader may wonder about the nature of the elements of $\mathrm{Cl}^v(V)$ for the valuation domain $V$ of Corollary 3.7. The clue comes from the proof of Theorem 3.5 and the following result.

**Proposition 3.8.** *Let $V$ be a valuation domain with quotient field $K$ and with maximal ideal $M$ that is idempotent and branched. Then for each nonprincipal $v$–invertible $v$–ideal $I$ of $V$, there exist two $M$–primary ideals $Q$ and $Q_1$ of $V$ and two elements $0 \neq x, y \in K$ such that $I = xQ$ and $I^{-1} = yQ_1$. Therefore* $\mathrm{Cl}^v(V) = \{[Q] \mid$ *$Q$ is an $M$–primary nonfinitely generated $v$–invertible $v$–ideal of $V\} \cup \{[V]\}$.*

*Proof.* As in the proof of Theorem 3.5, if $I$ is a nonfinitely generated $v$–invertible integral $v$–ideal of $V$, then $II^{-1} = M$. Since $M$ is branched, we must have $I = xQ$

for some $0 \neq x \in V$, where $Q$ is an $M$–primary ideal of $V$ [1, Theorem 2]. So $Q$ is a nonfinitely generated $v$–invertible $v$–ideal of $V$ and we have $Q^{-1} = y^{-1}H$, where $0 \neq y \in V$ and $H$ is obviously a nonfinitely generated $v$–invertible integral $v$–ideal of $V$. Another appeal to [1, Theorem 2] gives that $H = zQ_1$, where $Q_1$ is $M$–primary and $0 \neq z \in V$. Thus $[I] = [Q]$ and $[I^{-1}] = [x^{-1}Q^{-1}] = [(xy)^{-1}H] = [z(xy)^{-1}Q_1] = [Q_1]$, as claimed in the statement. The conclusion is now obvious once we note that for every nonzero finitely generated (or, equivalently, principal) fractional ideal $I$ of $V$, we have $[I] = [V]$, the identity element of $\mathrm{Cl}^v(V)$. □

In order to extend Example 3.2, we can start with the value group $G'$ of a valuation domain $V'$ of any dimension $1 \leq n \leq \infty$. If $G$ is a totally ordered additive subgroup of $\mathbb{R}$, we can construct the lexicographic direct sum $\Gamma := G' \oplus G$. The resulting group $\Gamma$ is a totally ordered abelian group upon which we can construct, using the Krull-Kaplansky-Jaffard-Heinzer-Ohm Theorem (see for instance [31, Corollary 18.5] or [43]), a valuation domain $D$ with a branched maximal ideal $M$ and a prime ideal $P$ directly below $M$ such that $D/P$ is a rank-one valuation domain with value group $G$ [31, page 223, Problem 20] and $D_P$ is a valuation domain with value group $G'$ [31, proof of Proposition (19.11)(3)]. An explicit example of this type is the following.

**Example 3.9.** *Let $V$ be a rank-one valuation domain with value group $G$ and let $K$ be the quotient field of $V$. Let $V'$ be an $n$–dimensional valuation domain, $1 \leq n \leq \infty$, with value group $G'$ such that the residue field of $V'$ is $K$ (note that this is possible by* [31, Corollary 18.5]*), and let $\pi' : V' \to K$ be the canonical projection. Then the pullback $D := \pi'^{-1}(V)$ is a valuation domain such that $D/P \cong V$ and $D_P = V'$, where $P := \pi'^{-1}(0)$. The value group of $D$ is the lexicographic direct sum $G' \oplus G$, $\dim(D) = n + 1$ (resp., $\infty$) if $n \neq \infty$ (resp., $n = \infty$), and $\mathrm{Cl}^v(D) = \mathbb{R}/G$.*

Arguing more or less as in Example 3.2, the properties listed in the above statement follow from [31, Proposition 18.2(3)], [12, Chapitre 6, §10, N. 2, Lemme 2], Theorem 2.7(2), and Theorem 3.5 (or, [27, Corollary 2.7]).

Note that in Example 3.2, if we set $S := V \setminus \{0\}$, then $D_S = K[\![X]\!]$ is PID. So $D$ is an example of a two-dimensional valuation domain such that $\mathrm{Cl}^v(D_S) = 0$, while $\mathrm{Cl}^v(D) \neq 0$. On the other hand, we can construct a valuation domain $D$ such that $\mathrm{Cl}^v(D) = 0$, but $\mathrm{Cl}^v(D_S) \neq 0$ for some ring of fractions $D_S$ of $D$. For instance, using the techniques of Examples 3.2 and 3.9, if we construct a valuation domain $D$ having as value group the lexicographic direct sum $\mathbb{Q} \oplus \mathbb{Z}$, then $D$ is a two-dimensional valuation domain with the height-one prime ideal $P$ of $D$ such that $D/P$ is a DVR (hence the maximal ideal of $D$ is principal) and $D_P$ is a rank-one valuation domain with value group $\mathbb{Q}$. Therefore $\mathrm{Cl}^v(D) = 0$ since every nonzero ideal of $D$ is divisorial [31, Exercise 12, page 431], but $\mathrm{Cl}^v(D_P) = \mathbb{R}/\mathbb{Q}$.

**Example 3.10.** **(1)** *Let $G$ be any abelian group. Then there is a quasilocal Krull domain $D$ with $\mathrm{Cl}^v(D) = G$ $(= \mathrm{Cl}^t(D)$* [31, Corollary 44.3]*) and $\mathrm{Pic}(D) = 0$.*

The first statement is due to L.G. Chouinard [15, Corollary 2]. It is obvious that $\mathrm{Pic}(D) = 0$ since an invertible ideal in a quasilocal domain is principal [31, page 72].

**(2)** *Let $D := K[X, Y]$, where $K$ is a field and $X, Y$ are two indeterminates over $K$. Then there are infinitely many distinct star operations on $D$ (more precisely, the cardinal number is $2^\alpha$, where $\alpha := \max\{|K|, \aleph_0\}$* [6, page 1639]*), and $\mathrm{Cl}^*(D) = 0$ for all star operations $*$ on $D$.*

Since $D$ is a UFD, thus $\mathrm{Cl}^v(D) = 0$ [31, Corollary 44.5]. The conclusion follows from the fact that $\mathrm{Cl}^*(D) \subseteq \mathrm{Cl}^v(D)$ for all star operations $*$ on $D$.

**Remark 3.11.** Recall from [28, p. 651] that an integral domain $D$ with a semistar operation $\star$ is an *H($\star$)–domain* if for each nonzero integral ideal $I$ of $D$ such that $I^\star = D^\star$, there exists a nonzero finitely generated ideal $J$ with $J \subseteq I$, such that $J^\star = D^\star$ (i.e., $I$ is $\star_f$–finite). When $\star = v$, the H($v$)–domains coincide with the H–domains introduced by Glaz and Vasconcelos [32, Remark 2.2 (c)].

It is obvious that every integral domain is an H($\star_f$)–domain, so the notion of H($\star$)–domain becomes interesting only when $\star$ is not of finite type.

Clearly a $\star$–Noetherian domain (i.e., an integral domain having the ascending chain condition on the quasi–$\star$–ideals [20, Section 3]) is an H($\star$)–domain [20, Lemma 3.3], thus we obtain in particular that Mori domains (or $v$-Noetherian domains; e.g. Noetherian and Krull domains) are H–domains. Houston and Zafrullah [38, Proposition 2.4] proved, more generally, that each *TV-domain* (i.e., an integral domain such that the $t$-operation coincides with the $v$–operation) is an H–domain. Conversely, a general class of H–domains which are not TV-domain was also given in [38].

It was shown in [59, Proposition 4.2] that an integral domain is an H–domain if and only if every $v$–invertible ideal is $t$–invertible. This statement can be easily generalized to the arbitrary star operation setting.

*Let $*$ be a star operation on an integral domain $D$. Then the following conditions are equivalent:*

(i) *$D$ is an H($*$)–domain (resp., an H–domain).*
(ii) *Each $*_f$–maximal ideal of $D$ is a $*$–ideal of $D$ (resp., each maximal $t$–ideal is a $v$–ideal).*
(iii) *For each $I \in \boldsymbol{F}(D)$, $I$ is $*$–invertible if and only if $I$ is $*_f$–invertible (resp., $I$ is $v$–invertible if and only if $I$ is $t$–invertible).*
(iv) *$\mathrm{Cl}^{*_f}(D) = \mathrm{Cl}^*(D)$ (resp., $\mathrm{Cl}^t(D) = \mathrm{Cl}^v(D)$).*

The equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) follow from [29, Proposition 11]. (iii)$\Rightarrow$(iv) Since a $*$–ideal is trivially also a $*_f$–ideal, thus under the assumption (iii), $\mathrm{Inv}^*(D) \subseteq \mathrm{Inv}^{*_f}(D)$. Therefore $\mathrm{Cl}^*(D) \subseteq \mathrm{Cl}^{*_f}(D)$. Since the reverse inclusion holds in general, we have $\mathrm{Cl}^*(D) = \mathrm{Cl}^{*_f}(D)$. (iv)$\Rightarrow$(iii) In this situation, $\mathrm{Inv}^{*_f}(D) = \mathrm{Inv}^*(D)$, i.e., for each $I \in \boldsymbol{F}(D)$, $I$ is a $*$–invertible $*$–ideal if and only if $I$ is a $*_f$–invertible $*_f$–ideal. Recall that if an ideal $I$ is $*$–invertible (resp., $*_f$–invertible), then $I^* = I^v$ (resp., $I^{*_f} = I^t$) (Proposition 2.2). Therefore, from the previous considerations we deduce that if $I$ is $*$–invertible, then it is $*_f$–invertible. The converse is trivial.

From the previous remark, we deduce immediately the following two corollaries. The first one generalizes Corollary 3.6 (cf. also Proposition 3.1 (2b)).

**Corollary 3.12.** *Let $D$ be a quasilocal integral domain with principal maximal ideal $M$. Then* $\mathrm{Cl}^*(D) = 0$ *for all star operations* $*$ *on* $D$.

*Proof.* By Remark 3.11 ((ii)⇒(i)), since $M$ is principal, $D$ is an $\mathtt{H}$–domain, and since $D$ is quasi–local, $\mathrm{Inv}(D) = \mathrm{Prin}(D)$, and so $\mathrm{Cl}^t(D) = 0$. Therefore $\mathrm{Cl}^*(D) = \mathrm{Cl}^t(D) = \mathrm{Cl}^v(D) = 0$ for all star operations $*$. □

The next corollary generalizes Example 3.10 (2).

**Corollary 3.13.** *If $D$ is a pre-Schreier domain and an $\mathtt{H}$–domain (e.g., if $D$ is a UFD), then* $\mathrm{Cl}^*(D) = 0$ *for all star operations* $*$ *on* $D$.

*Proof.* We have already observed above that $\mathrm{Cl}^t(D) = 0$ for a pre-Schreier domain $D$ [14, Proposition 1.4]. The parenthetical statement follows from the fact that a GCD-domain is a pre-Schreier domain, and a UFD is a GCD-domain and a Krull domain. □

We have seen that if $V$ is a valuation domain with principal maximal ideal $M$, then $\mathrm{Cl}^v(V) = 0$ (Proposition 3.1 (2b)), and we have found out that if the maximal ideal $M$ is idempotent and branched, then $\mathrm{Cl}^v(V)$ is isomorphic to $\mathbb{R}/G$, where $G$ is the value group of a certain nondiscrete rank-one valuation domain (Corollary 3.7). We also know that if the maximal ideal $M$ of $V$ is not idempotent then $M$ is principal. This leaves us with the case of $V$ with $M$ unbranched (and thus necessarily idempotent [31, Theorem 17.3 (b)]). At present we know very little about this case, but an example in [4, Example 1] gives a valuation domain $V$ with unbranched maximal ideal $M$ such that $V$ affords a nonfinitely generated $v$–ideal $I$ with $II^{-1} = M$. Consequently, a valuation domain with unbranched maximal ideal may have nonzero $v$–class group.

We end the paper with some observations in the not necessarily quasilocal setting, but related to the valuation domain case. Recall that Bouvier [13, Proposition 2] proved that $D$ is a GCD-domain if and only if $D$ is a P$v$MD and $\mathrm{Cl}^t(D) = 0$ and, moreover, in [14, Proposition 1.4] Bouvier and Zafrullah have shown that if $D$ is a pre-Schreier domain, then $\mathrm{Cl}^t(D) = 0$. An interested reader may want to know if results similar to these hold for $\mathrm{Cl}^v(D)$. It appears that one answer may suffice for both questions. Because $\mathrm{Cl}^t(D) \subseteq \mathrm{Cl}^v(D)$, and so if $D$ is a P$v$MD with $\mathrm{Cl}^v(D) = 0$, then $D$ is a GCD-domain with a slight difference. The difference is that not every GCD-domain $D$ has $\mathrm{Cl}^v(D) = 0$. One example comes from Theorem 2.7 and another slightly more general example follows from Corollary 3.7. Indeed, as every GCD-domain is also *a Schreier domain* (i.e., an integrally closed domain in which every element is primal in Cohn's sense [16]), and hence a pre-Schreier domain [18], and as a P$v$MD is pre-Schreier (or, Schreier) if and only if it is a GCD-domain [14, Corollary 1.5], we conclude that for a pre-Schreier domain $D$ it is not necessary that $\mathrm{Cl}^v(D) = 0$ (for an explicit example of this type cf. for instance [16]). We can however make a somewhat more general statement in this connection. For this, recall that an integral domain $D$ is a *$v$–domain* if every nonzero finitely generated ideal of $D$ is $v$–invertible.

**Proposition 3.14.** *Let $D$ be a $v$–domain (e.g., a completely integrally closed integral domain* [31, Theorem 34.3]*). If $\mathrm{Cl}^v(D) = 0$, then $D$ is a GCD-domain, but not conversely.*

*Proof.* Let $I$ be a nonzero finitely generated ideal of $D$. Then $(II^{-1})^v = D$. So $I^{-1}$, being a $v$–invertible $v$–ideal, must be principal because $\mathrm{Cl}^v(D) = 0$. So for every nonzero finitely generated ideal $I$ of $D$, we have that $I^v = (I^{-1})^{-1}$ is principal, which makes $D$ a GCD-domain. That the converse is not true follows from comments prior to this proposition. □

**Remark 3.15.** (1) The previous, not very striking, statement gives us some interesting candidates for the study of the $v$–class group.

(a) Nagata (in [45] and [46]) gave an example of a completely integrally closed one-dimensional quasilocal integral domain $D$ that is not a valuation domain. Obviously $D$ is not a GCD-domain (since, as recalled above, a GCD-domain is a particular P$v$MD). So by Proposition 3.14, $\mathrm{Cl}^v(D) \neq 0$. It would be of interest to find $\mathrm{Cl}^v(D)$ in this case.

(b) Let $k$ be a field, let $Y, Z, X_1, X_2, \dots, X_n, \dots$ be indeterminates, set $K := k(Y, Z, X_1, X_2, \dots, X_n, \dots)$ and $R := k(X_1, X_2, \dots, X_n, \dots)[Y, Z]_{(Y,Z)}$. Inside the field $K$, Heinzer and Ohm [36, Section 2] and Mott and Zafrullah [44, Example 2.1] consider a domain $D$ which is not a P$v$MD. This domain $D$ is the intersection of the regular local ring $R$ with a denumerable family of discrete valuation domains of rank one in $K$, and so $D$ is completely integrally closed. Not being a P$v$MD makes $D$ non-GCD, and so $\mathrm{Cl}^v(D) \neq 0$. Again, it would be of interest to know $\mathrm{Cl}^v(D)$.

The main question in both cases is: must $\mathrm{Cl}^v(D)$ be a homomorphic image of $(\mathbb{R}, +)$ as we saw in the valuation domain cases?

(2) Our study of $v$–class groups appears to raise a lot of other questions. We mention some of those questions here.

- What is $\mathrm{Cl}^v(D[X])$ in terms of $\mathrm{Cl}^v(D)$?
- Halter-Koch in [34, pages 167–185] talks about valuation monoids. Consider $\mathcal{S}$ a valuation monoid under addition. Defining the $v$–operation on $\mathcal{S}$ and $\mathrm{Cl}^v(\mathcal{S})$ in the obvious fashion, find analogs of Theorems 2.7 and 3.5. Also, study $\mathrm{Cl}^v(D[\mathcal{S}])$ when $\mathrm{Cl}^v(D)$ is known. In particular, find $\mathrm{Cl}^v(D[\mathbb{Q}^+])$ in terms of $\mathrm{Cl}^v(D)$.

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