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COMMENTS ON UNIQUE FACTORIZATION IN NON-UNIQUE FACTORIZATION DOMAINS

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Dedicated to the memory of Paul Cohn

Abstract. Let D be an integral domain throughout. Call two elements $x, y \in D\backslash\{0\}$ v-coprime if $xD \cap yD = xyD$. Call a nonzero non unit r of an integral domain D rigid if for all $x, y|r$ we have $x|y$ or $y|x$. Also call D semirigid if every nonzero non unit of D is expressible as a finite product of rigid elements. We show that a semirigid domain D is a GCD domain if and only if D satisfies $*$: product of every pair of non-v-coprime rigid elements is again rigid. This research links the ancient notion of semirigid GCD domains of [Manuscripta Math. 17(1975), 55-66] with the recent work in [J. Pure Appl. Algebra 224 (12) (2020), 106430], in the context of "Unique factorization property of non-unique factorization domains."

1. Introduction

Let D be an integral domain with quotient field K. Some recently introduced concepts may remind one of some old and some recent concepts. These are concepts such as a homogeneous element of Chang [11], one that belongs to a unique maximal t-ideal and a valuation element of Chang and Reinhart [14], i.e. an element a such that $aV \cap D = aD$ for some valuation ring V with $D \subseteq V \subseteq K$.

Chang [11] calls a domain a HoFD if every nonzero non unit of D is expressible as a product of mutually t-comaximal homogeneous elements and says HoFDs were first studied in [5], of course with different terminology. (A homogeneou element was "t-pure" and a HoFD was semi t-pure.) According to [14, Corollary 1.2] a valuation element a of a domain D has the property that for all $x, y|a$ we have $x|y$ or $y|x$. This makes a a rigid element of Cohn [13]. To be exact, let's call an element r of D rigid if r is a nonzero non-unit such that for all $x, y|r$ we have $x|y$ or $y|x$. Let's also call D semirigid (resp., semi homogeneous) if every nonzero non unit of D is expressible as a finite product of rigid (resp., homogeneous) elements of D. The trouble with the semirigid (resp., semi homogeneous) domains is that they are very general. For example every irreducible element, i.e, a nonzero non-unit a such that $a = \alpha\beta \Rightarrow \alpha$ is a unit or β is, is rigid. But the atomic domains, i.e, domains whose nonzero non-units are expressible as finite products of irreducible elements, often have little or no form of uniqueness of factorization [2]. For example, in $D = F[[X^2, X^3]]$, that is Noetherian and hence atomic, the elements X^2 and X^3 are irreducible, and

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$(X^2)^3 = (X^3)^2 = X^6$. That is X^6 has two distinct factorizations. On the other hand, as we shall show, semi homogeneous domains are actually HoFDs and so do have a sort of uniqueness of factorization, but only just.

One way of getting such wayward concepts to deliver unique factorization of some sort is to bring in a somewhat stronger notion of coprimality and some conditions. Call two elements a, b v-coprime if $aD \cap bD = abD$. Obviously a, b are v-coprime if and only if $(a, b)^{-1} = D$, if and only, if $((a, b)^{-1})^{-1} = (a, b)_v = D$, where $A \mapsto A_v = (A^{-1})^{-1}$ is the usual star operation called the v-operation on $F(D)$, the set of nonzero factional ideals of D. The notion of v-coprimality has been discussed in detail in [18], where it is shown, in somewhat general terms, that if, for $a, b, c \in D \backslash \{0\}$, $(a, b)_v = D$ and $a|bc$, then $a|c$. It was also shown in [18] that for $r_1, ..., r_n, x \in D \backslash \{0\}$ $(r_1...r_n, x)_v = D$ if and only if $(r_i, x)_v = D$. Let's call two homogeneous (resp., rigid) elements a, b similar, denoted $a\tilde{\ }b$, if $(a, b)_v \neq D$. We plan to show that a semi homogeneous domain is a "HoFD" because the product of every pair of similar homogeneous elements of D is again a homogeneous element, of D, similar to them. We also show that a semirigid domain is a semirigid GCD domain if and only if the product of each pair of non-v-coprime rigid elements is rigid and give examples to show that the product of two non-v-coprime rigid elements may not be rigid. Incidentally semirigid GCD domains were first studied in [20] and are precisely the so called UVFDs of [14]. We shall also give examples to show that a homogeneous element may not be rigid and a rigid element may not be homogeneous.

It seems best to give the reader an idea of the v- and the t-operations and some related concepts that we shall have the occasion to use. For $I \in F(D)$, the set $I^{-1} = \{x \in K | xI \subseteq D\}$ is again a fractional ideal and thus the relation v: $I \mapsto I_v$ is a function on $F(D)$. This function is called the v-operation on D. Similarly the relation t: $I \mapsto I_t = \cup\{F_v | \ 0 \neq F$ is a finitely generated subideal of $I\}$ is a function on $F(D)$ and is called the t-operation on D. These are examples of the so called star operations. The reader may consult Jesse Elliott's book [15], for these operations. A fractional ideal I is called a v-ideal (resp. a t-ideal) if $I_v = I$ (resp., $I_t = I$).The rather peculiar definition of the t-operation allows one to use Zorn's Lemma to prove that each integral domain that is not a field has at least one integral t-ideal maximal among integral t-ideals. This maximal t-ideal is prime and every proper, integral t-ideal is contained in at least one maximal t-ideal. A minimal prime of a t-ideal is a t-ideal and thus every height one prime is a t-ideal. The set of all maximal t-ideals of a domain D is denoted by $t\text{-}Max(D)$. It can be shown that $D = \cap_{M \in t\text{-}Max(D)} D_M$. A fractional ideal I is said to be t-invertible if $(II^{-1})_t = D$. A domain in which every nonzero finitely generated ideal is t-invertible is called a Prufer v-multiplication domain (PVMD), a Prufer domain is a PVMD with every nonzero ideal a t-ideal. Griffin [16] showed that D is a PVMD if and only if D_M is a valuation domain for each maximal t-ideal M of D. Given any domain D the set $t\text{-}inv(D)$ of all t-invertible fractional t-ideals of D is a group under the t-operation $(I \times_t J = (IJ)_t$. The group $t\text{-}inv(D)$ has the group $P(D)$ of nonzero principal fractional ideals as its subgroup. The t-class group of D is the quotient group $Cl_t(D) = t\text{-}inv(D)/P(D)$. What makes this group interesting is that if D is a Krull domain $Cl_t(D)$ is the divisor class group and if D is a Prufer domain, $Cl_t(D)$ is the ideal class group. Of interest for this note is the fact that a PVMD D is a GCD domain if and only if $Cl_t(D)$ is trivial. This group was introduced in [10].

Next call an element $a \in D\backslash\{0\}$ primal if for all $b, c \in D\backslash\{0\}$ $a|bc$ implies that $a = rs$ where $r|b$ and $s|c$. A domain all of whose nonzero elements are primal is called a pre-Schreier domain and an integrally closed pre-Schreier domain was called a Schreier domain in [12]. Note that if D is pre-Schreier then $Cl_t(D)$ is trivial, [10]. Call a nonzero element p of D completely primal if every factor of p is again primal. A prime element is an example of a primal element. According to Cohn [12], if S is a set multiplicatively generated by completely primal elements of an integrally closed domain D such that D_S is a Schreier domain, then D is a Schreier domain. This Theorem is usually referred to as: Cohn's Nagata type Theorem for Schreier domains.

2. Results

Let's note that for a finitely generated nonzero ideal $I = (x_1, ..., x_n)$ we have $I_v = I_t$, so $x_1, ..., x_n$ being v-coprime (i.e., $(x_1, ..., x_n)_v = D$) is the same as $x_1, ..., x_n$ being t-comaximal (i.e., $(x_1, ..., x_n)_t = D$), which boils down to: $x_1, ..., x_n$ do not share any maximal t-ideal. We also note that a is a homogeneous element if aD is a t-homogeneous ideal in the sense of [7] and, sort of, following the convention of [7] we shall denote by $M(a)$ the maximal t-ideal containing the homogeneous element a. Indeed we have $M(a) = \{x \in D| \ (x, a)_v \neq D\}$ (cf. [7, (2) Proposition 1]).

Lemma 2.1. *Let a and b be two homogeneous elements of D then $(a, b)_v \neq D$ if and only if (a, b) is contained in the same maximal t-ideal if and only if ab is a homogeneous element.*

Proof. Let b be a homogeneous element belonging to the maximal t-ideal P. For any nonzero finitely generated ideal A, $(A, b)_v \neq D$ implies that that $A \subseteq P$. This is because $(A, b)_v \neq D$ implies (A, b) has to be contained in some maximal t-ideal and P is the only maximal t-ideal that contains b. So $A \subseteq P$. Now $(a, b)_v \neq D$ implies that a, b both belong to the same maximal t-ideal say P. Next note that $x \in M(a) \Leftrightarrow (x, a)_v \neq D$. So $x \in M(a)$ implies x belongs to P. Thus $M(a) = P$ and similarly $M(b) = P$ forcing $M(a) = M(b)$. Suppose ab belongs to a maximal t-ideal P. Then $a \in P$ or $b \in P$. If $a \in P$, then $M(a) = P$. But as $(b, a)_v \neq D$, $M(a) = M(b)$ whence ab is a homogeneous element, as $P(a) = P(b)$ is the only maximal t-ideal containing ab. Finally if ab is t-homogeneous then, by definition, $(a, b)_v \neq D$. □

Proposition 1. *An integral domain D is a HoFD if and only if D is a semi homogeneous domain.*

Proof. Suppose that D is a semi homogeneous domain. Lemma 2.1 shows that the product of every pair of similar homogeneous elements of D is homogeneous. Let $x = h_1 h_2 ... h_n$ where each of h_i is a homogeneous element. Now $M_1, ...M_r$ be the set of distinct maximal t-ideals containing h. Let $H_j = \Pi h$ where h ranges over $h_i \in M_j$. By Lemma 2.1, H_j are homogeneous and mutually t-comaximal. Thus we have $x = \Pi_{j=1}^{r} H_j$ where H_i are mutually v-coprime homogeneous. The converse is obvious. □

It was shown in [20] that if a nonzero non unit x in a GCD domain is expressible as a finite product of rigid elements then x is uniquely expressible as a product of finitely many mutually coprime rigid elements. Thus showing that in a semirigid GCD domain, every nonzero non unit x is expressible uniquely as a product of mutually coprime elements. So a valuation ring V of any rank is an example of a semirigid GCD domain and so is a polynomial ring over V. Griffin, in [17], called a domain D an Independent Ring of Krull type (IRKT) if D has a family of prime ideals $\{P_\alpha\}_{\alpha\in I}$ such that (a) D_{P_α} is a valuation domain for each $\alpha \in I$, (b) $D = \cap_{\alpha\in I} D_{P_\alpha}$ is locally finite and (c) No pair of distinct members of $\{P_\alpha\}_{\alpha\in I}$ contains a nonzero prime ideal. It was shown in Theorem 5 of [20] that a semirigid GCD domain is indeed an IRKT. Also, it was shown in Theorem B of [19] that a GCD IRKT is a semirigid GCD domain. Later, a domain satisfying only (b) and (c) above, requiring that P_α are maximal t-ideals, was called in [9] a weakly Matlis domain. An IRKT is a PVMD, [17]. Also, noting that a GCD domain is a PVMD which makes localization at each maximal t-ideal a valuation domain we have each of D_{P_α} a valuation domain, in the definition of a weakly Matlis domain and making it an IRKT. Finally, a GCD IRKT is a semirigid GCD domain, by Theorem B of [19].

Lemma 2.2. *Let D be a semirigid domain with $*$: for every pair r, s of rigid elements $(r, s)_v \neq D \Leftrightarrow rs$ is rigid. Then the following hold. (1) Given that r, s are two similar rigid elements. Then r and s are comparable, i.e., $r|s$ or $s|r$, (2) If r is a rigid element and s, t are rigid elements, each similar to r, then s and t are similar, (3) A finite product of mutually similar rigid elements is rigid similar to each of the factors and (4) if a rigid element r divides a product $x = x_1x_2...x_n$ of mutually v-coprime rigid elements $x_1, ..., x_n$ then r divides exactly one of the x_i, in a semirigid domain with property $*$.*

Proof. (1) Straightforward, as rs is rigid, (2) $r|s$ or $s|r$ and $r|t$ or $t|r$. Four cases arise (i) $r|s$ and $r|t \Rightarrow (s, t)_v \neq D$, (ii) $r|s$ and $t|r \Rightarrow t|s$ (iii) $s|r$ and $r|t \Rightarrow s|t$ (iv) $s|r$ and $t|r \Rightarrow s|t$ or $t|s$. In each case we have $s\tilde{\ }t$. (3) Suppose that D is semirigid with the given property $(*)$. Using induction, one can show that in a semirigid domain with $(*)$, a finite product of mutually similar rigid elements is rigid. This is how it can be accomplished. We know that the product of any two similar rigid elements is rigid. Assume that we have established that the product of any set of n of rigid elements, $r_1, r_2, ..., r_n$, similar to one of them and hence, by (2), similar to each other, is rigid. Let $\boldsymbol{r} = r_1r_2...r_n$ and let s be a rigid element similar to, one and hence, each of r_i and hence to $\mathbf{r}$. But then by $*$, $\mathbf{r}s$ is rigid. Finally for (4) note that $(r, x)_v = rD \neq D$, because $r|x$. So r cannot be v-coprime to each of x_i. Now, say, r is non-v-coprime to x_i, x_j for $i \neq j$. But then, by (2), $x_i\tilde{\ }x_j$ which is impossible because $(x_i, x_j)_v = D$. So r is non-v-coprime to exactly one of x_i, say x_k Now as D has the property $*$ and as r and x_k are rigid, one of them divides the other. But since $r|x$ already, we conclude that $r|x_k$. □

Theorem 2.3. *Let D be a semirigid domain. Then every nonzero non unit of D is either rigid or can be written uniquely as a product of finitely many mutually v-coprime rigid elements if and only if $*$: for every pair r, s of rigid elements $(r, s)_v \neq D \Leftrightarrow rs$ is rigid holds.*

Proof. Let $x = r_1r_2...r_n$ be a nonzero non unit of D. Pick r_1 and collect all the rigid factors, from r_i, $(i = 1, ..., n)$, that are similar to r_1. Next suppose that by a relabeling we can write $x = r_1r_2...r_{s_1}r_{s_1+1}...r_n$ where r_i $(i = 1, ..., s_1)$ are all the rigid factors of x that are similar to r_1. Set $\mathbf{r}_1 = r_1r_2...r_{s_1}$. Note that since, by the procedure, each of r_i $(i = 1, ..., s_1)$ is v-coprime to each of r_{i_1} $(i_1 = s_1 + 1, ..., n)$ we conclude that $\mathbf{r}_1$ is v-coprime to each of r_{i_1} $(i_1 = s_1 + 1, ..., n)$ and, of course, each of r_{i_1} $(i_1 = s_1 + 1, ..., n)$ v-coprime to $\mathbf{r}_1$. Now select all the rigid elements similar to r_{s_1+1} and suppose that by a relabeling we can write $r_{s_1+1}...r_n = r_{s_1+1}r_{s_1+2}...r_{s_2}...r_n$, where r_j $(j = s_1 + 1, ..., s_2)$ are similar to r_{s_1+1}. Set $\mathbf{r}_2 = r_{s_1+1}r_{s_1+2}...r_{s_2}$. By, Lemma 2.2, $\mathbf{r}_2$ is rigid. Since $r_{s_1+1}, r_{s_1+2}, ..., r_{s_2}$ are each v-coprime to $\mathbf{r}_1$, and so is their product, we conclude that $\mathbf{r}_1$ and $\mathbf{r}_2$ are v-coprime rigid elements. Thus $x = \mathbf{r}_1\mathbf{r}_2 r_{s_2+1}...r_n$ and continuing this manner we can write $x = \mathbf{r}_1\mathbf{r}_2...\mathbf{r}_m$ where $\mathbf{r}_i$ are mutually v-coprime rigid elements.

Now let $x = \mathbf{r}_1\mathbf{r}_2...\mathbf{r}_m$ be a product of mutually v-coprime rigid elements in a domain D with property $*$. Also let $x = \mathbf{s}_1\mathbf{s}_2...\mathbf{s}_n$. We claim that each of the $\mathbf{r}_i$ is an associate of exactly one of the $\mathbf{s}_j$ and hence $m = n$. For this note that by (4) of Lemma 2.2, $\mathbf{r}_1|\mathbf{s}_1\mathbf{s}_2...\mathbf{s}_n$ implies that $\mathbf{r}_1$ divides exactly one one of the $\mathbf{s}_j$, say $\mathbf{s}_1$, by a relabeling. But then, considering $\mathbf{s}_1|\ \mathbf{r}_1\mathbf{r}_2...\mathbf{r}$ and noting that $\mathbf{s}_1\tilde{}\mathbf{r}_1$ we conclude that $\mathbf{s}_1|\mathbf{r}_1$. This leaves us with $\mathbf{r}_2...\mathbf{r}_m = \mathbf{s}_2...\mathbf{s}_n$. Now eliminating one by one, in this manner, and noting that $\mathbf{r}_i$, $\mathbf{s}_j$ are non units will take us to the conclusion, eventually.

Conversely let D be a semirigid domain in which every nonzero non unit is either a rigid element or is uniquely expressible as a finite product of mutually v-coprime rigid elements and consider $x = rs$ where r and s are any two similar rigid elements. If in each case rs is rigid, we are done. To ensure that there is no other possibility we proceed as follows. Because x is expressible as a product of finitely many mutually v-coprime rigid elements we can write $rs = r_1...r_n$. By (4) of Lemma 2.2, each of r, s divides exactly one of the r_i, so $n \leq 2$. So let $rs = r_1r_2$, where $r|r_1$ and $s|r_2$. Now this too is impossible because, by assumption, r and s are non-v-coprime while r_1 and r_2 are v-coprime and obviously a pair of v-coprime elements (such as r_1, r_2) cannot have factors like $r|\acute{r}_1$ and $s|r_2$ with $(r, s)_v \neq D$. (For this note that $r|\acute{r}_1$ and $s|r_2$ implies that $(\acute{r}_1, r_2) \subseteq (r, s)$ forces $D = (\acute{r}_1, r_2)_v \subseteq (r, s)_v \neq D$, which is a contradiction.) □

Corollary 1. *A semirigid domain with property $*$ is a GCD domain.*

Proof. We shall show that $xD \cap yD$ is a principal ideal for each pair x, y of nonzero elements of D. Indeed if any of x, y is a unit then $(x, y)_v = D$ a principal ideal. But $(x, y)_v = D \Leftrightarrow xD \cap yD = xyD$. So let's assume that both x and y are non units. Next let, say, x be a rigid element and y a general element. Then $y = r_1...r_n$ where r_i are mutually v-coprime rigid elements. If x is v-coprime to each of r_i then $(x, y)_v = D$, and so $xD \cap yD$ is principal. If on the other hand $(x, y)_v \neq D$, then x is non-v-coprime to exactly one of r_i. (For otherwise, x non-v-coprime to r_i, r_j for $i \neq j$ would imply, by Lemma 2.2, that $r_i\tilde{}r_j$ which is impossible because r_i and r_j are v-coprime.) Suppose that, by a relabeling x is non-v-coprime to r_1. Then $x|r_1$ or $r_1|x$. This gives $xD \cap yD = (xD \cap r_1D) \cap r_2D \cap ... \cap r_nD = r_1D \cap r_2D \cap ... \cap r_nD$ in case $x|r_1$, giving us $xD \cap yD = yD$. If on the other hand $r_1|x$, $(xD \cap r_1D) = xD$ and x is v-coprime to each of $r_2, ..., r_n$. Thus giving $xD \cap yD = xD \cap r_2D \cap ... \cap r_nD = xr_2....r_nD$, a principal ideal.

This leaves the case when $x = a_1...a_rb_1...b_s$ and $y = c_1c_2...c_td_1...d_u$ are non units, each a product of mutually v-coprime rigid elements. (Note that $a_1, ..., a_r, b_1, ..., b_s$ are mutually v-coprime in case of x and $c_{1,}c_2, ...c_t, d_1, ..., d_s$ are mutually v-coprime in case of y.) We can write $x = ab$ where $a = a_1...a_r$ is the product of rigid elements a_i that are v-coprime to each of the, mutually v-coprime, rigid factors of y and

$b = b_1...b_s$ is the product of those rigid factors b_j each of which is non-v-coprime to y. Obviously a and b are v-coprime. Similarly we have $y = cd$ where $c = c_1c_2...c_t$ is the product of all those rigid elements that are v-coprime to every rigid factor of x while $d = d_1...d_u$ is the product of those rigid elements that are non-v-coprime to x. Obviously $u = s$ and each of d_i is similar to exactly one of b_i, say b_j. Here too c and d are v-coprime. Here, by a relabeling, we can assume that $b_i\tilde{}d_i$ for $i = 1, ..., s$.

Now consider $xD \cap yD = abD \cap cdD = aD \cap bD \cap dD \cap cD$ (because $(a, b)_v = D = (c, d)_v$). Now as, by their descriptions, aD and cD are v-coprime to each other and to bD and dD, we have $xD \cap yD = acD \cap bD \cap dD$. Next $bD \cap dD = \cap_{i=1}^{i=s} b_iD \cap (\cap_{i=1}^{i=s} d_iD) = \cap_{i=1}^{i=s}(b_iD \cap d_iD)$. Since by assumption $b_i\tilde{}d_i$ for $i = 1, ..., s$ $b_i|d_i$ or $d_i|b_i$ and so $(b_iD \cap d_iD) = d_iD$ if $b_i|d_i$ and $(b_iD \cap d_iD) = b_iD$ if $d_i|b_i$. In short, in each case $(b_iD \cap d_iD)$ is a principal ideal. Since all of $b_1, ..., b_s$ (resp., all of $d_1, ..., d_s$) are mutually v-coprime, all of $(b_iD \cap d_iD)$ are mutually v-coprime. Consequently $bD \cap dD = \cap_{i=1}^{i=s}(b_iD \cap d_iD)$ is an intersection of mutually v-coprime principal ideals and hence is principal. Say $bD \cap dD = \cap_{i=1}^{i=s}(b_iD \cap d_iD) = hD$. Thus $xD \cap yD = acD \cap bD \cap dD = acD \cap hD$. But as ac is v-coprime to each of the mutually v-coprime rigid factors of h and hence to h we have $acD \cap hD$ principal. But then $xD \cap yD$ is principal. Thus, in all possible cases, we have established that in a semirigid domain D with property $*$, $xD \cap yD$ is principal for each pair x, y of nonzero elements of D. This establishes that D is a GCD domain. □

In Proposition 2.1 the authors of [14] show that a VFD is a Schreier domain.

Corollary 2. *For an integral domain D the following are equivalent.*

(1) D is a semirigid domain with property $*$,
(2) D is a semirigid GCD domain,
(3) D is a semi homogeneous GCD domain,
(4) D is a HoFD PVMD ,
(5) D is a weakly Matlis GCD domain,
(6) D is a GCD VFD.
(7) D is a UVFD,
(8) D is a VFD such that product of every pair of non-coprime valuation elements is again a valuation element.
(9) D is a pre-Schreier semirigid domain with property $*$.

Proof. (1) $\Rightarrow$ (2) follows from Corollary 1, (2) $\Rightarrow$ (3) let r be a rigid element of a GCD domain D and consider $P(r) = \{x \in D | (x, r)_v \neq D\}$. By [20, Lemma 1] $P(r)$ is a prime ideal. To see that $P(r)$ is a t-ideal note that because D is a GCD domain, $(x, r)_v \neq D \Leftrightarrow x = a\rho$ where ρ is a non unit factor of r. Thus $x_1, x_2, ..., x_n \in P(r) \Rightarrow (x_1, x_2, ..., x_n) \subseteq (\rho_1)$ where ρ_1 is a non unit factor of r. But then, $x_1, x_2, ..., x_n \in P(r) \Rightarrow (x_1, x_2, ..., x_n)_v \subseteq P(r)$. Thus $P(r)$ is a t-ideal. Finally, let M be a prime ideal properly containing $P(r)$ and let $y \in M \backslash P(r)$. By the definition of $P(r)$, $(y, r)_v = D$. But then $M_t = D$ and this shows that $P(r)$ is actually a maximal t-ideal. Finally, using the definition, it can be easily established that for any pair of rigid elements r, s of D, $P(r) = P(s)$ if and only if $r\tilde{}s$. Thus every rigid element in a GCD domain D, belongs to a unique maximal t-ideal and hence is a homogeneous element. Consequently a semirigid GCD domain is a semi homogeneous GCD domain. For (3) $\Rightarrow$ (4) we proceed as follows. Note that a semi homogeneous domain is a HoFD, by Proposition 1 and, it is well known that, a GCD domain is a PVMD. Next (4) $\Rightarrow$ (5), since a HoFD D is a weakly Matlis domain with $Cl_t(D) = 0$ [11, Theorem 2.2] and since a PVMD D with $Cl_t(D) = 0$ is a GCD domain [10, Proposition 2], we conclude that a PVMD HoFD is a weakly Matlis GCD domain. Now for (5) ⇒ (6), note that Corollary 4.5 of [14] says that a UVFD is a weakly Matlis GCD domain and as a UVFD is a VFD, in particular, we have the conclusion. Next, (6) ⇒ (7) follows because, according to Theorem 4.2 of [14], a PVMD VFD is a UVFD and so a GCD VFD is a UVFD. For (7) ⇒ (8), let D be a UVFD then D is a VFD. Take two non-coprime valuation elements u, v and using the fact that uv is an element of a UVFD write $uv = a_1a_2...a_n$ where a_i are mutually incomparable and hence mutually coprime. Now $u|a_1a_2...a_n$ implies that $u = u_1u_2...u_n$ where $u_i|a_i$, because a VFD is Schreier [14], see also Cohn [12]. We claim that exactly one of the u_i is non unit. For if say u_1 and u_2 are non units then being factors of coprime elements u_1 and u_2 are incomparable and this contradicts the fact that u is a valuation element (cf. [14, (2) of Corollary 1.2]]). So u divides exactly one of the a_i. Similarly v divides exactly one of the a_i. Next u and v cannot divide two distinct a_i for that would make u, v coprime, which they are not. Now suppose that $u|a_1$. Then $v = (a_1/u)a_2...a_n$ and v cannot divide any of $a_2, ..., a_n$ as that would make v coprime with u. So, v must divide (a_1/u). But then $1 = (a_1/uv)a_2...a_n$, forcing $uv = a_1$ and forcing the conclusion that in the VFD D the product of any pair of non coprime valuation elements is again a valuation element. For (8) ⇒ (9) note that as each valuation element is rigid, a VFD is semirigid. Since a VFD is Schreier we can say that D is a pre-Schreier semirigid domain. Also the product of two non coprime valuation elements being a valuation element translates to the product of two non-v-coprime rigid elements is rigid and that is the property $*$. Finally (9) ⇒ (1) is direct. □

While Corollary 2 establishes that the most ancient concept of semirigid GCD domains of [20] is precisely the most modern concept of UVFDs of [14], it raises the following question.

Question Must a Schreier semirigid domain be a semirigid GCD domain?

This question becomes interesting in view of the fact that the authors of [14] ask a similar question: is a VFD a semirigid GCD domain? (Actually they ask: Is a VFD a weakly Matlis GCD domain?) The other point of interest is that, according to [12] an atomic Schreier domain is a UFD. In fact, once we recall necessary terminology, we have the following more general result.

Proposition 2. *(cf.,* [8, [8]], *Proposition 3.2]). In an integral domain with PSP property, every atom is a prime. Consequently an atomic domain with PSP property is a UFD.*

Here a polynomial $f(X) = \sum_{i=0}^{i=n} a_iX^i$ is primitive if the coefficients a_i of f have no non unit common factor and f is super primitive if the coefficients $a_0, ..., a_n$ are v-coprime. Also a domain D has the PSP property if every primitive polynomial is super primitive. Now as was indicated in [8, [8]] a domain with PSP property is much more general than a pre-Schreier domain. Lest hopes run too high, we hasten to offer the following example of a Schreier domain in which the product of two rigid elements is not rigid.

Example 2.4. Let $\mathbb{Z}$ denote the ring of integers, let $\mathbb{Q}$ be the ring of rational numbers and let X, Y be two indeterminates over $\mathbb{Q}$. Construct the two dimensional regular local ring $R = \mathbb{Q}[[X, Y]]$ and for p a prime element set $D =$

$\mathbb{Z}_{(p)} + (X,Y)\mathbb{Q}[[X,Y]]$. This ring D is a Schreier domain with two rigid elements X, Y such that $(X,Y)_v \neq D$, yet XY is not rigid.

Illustration: Indeed D is a quasi local ring with maximal ideal principal and of course X and Y are divisible by every power of p. That D is integrally closed follows from the fact that $D \subseteq \mathbb{Q}[[X,Y]]$ which is integrally closed, that $\mathbb{Z}_{(p)}$ is integrally closed and that X and Y are divisible by powers of p. For D being Schreier let S be multiplicatively generated by p. Then S is generated by completely primal elements and D_S is a UFD. Hence D is Schreier, by Cohn's Nagata type Theorem. Now look at X. Every factor of X of the form p^r or X/p^s. So any pair of factors of X is one of the forms: $(p^r, p^s), (p^r, X/p^s), (X/p^r, X/p^s)$, $r, s \geq 0$, and in each case one divides the other. Same with Y. Now $(X,Y)_v \neq D$, because $p|X, Y$. So X and Y are non-v-coprime rigid elements of D. Yet XY cannot be because X does not divide Y.

Remark 2.5. The above example has often appeared, in various guises, in papers in which Dan Anderson and I have been coauthors, see e.g. [5, page 344], [9].

It was shown in the proof of (2) $\Rightarrow$ (3) of Corollary 2 that a rigid element is a homogeneous element. However a rigid element may not generally be a homogeneous element. For an atom is rigid but, say, in a Krull domain with torsion divisor class group an atom can be in more than one height one prime ideals which can be shown to be maximal t-ideals. (For a concrete example $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain in which 3 is well known to be an irreducible element, but $(3) = (3, 1-2\sqrt{-5})(3, 1+2\sqrt{-5})$ where $(3, 1-2\sqrt{-5}), ((3, 1+2\sqrt{-5})$ are height one prime ideals and hence maximal (t-) ideals of $\mathbb{Z}[\sqrt{-5}]$. (Recall here that a Dedekind domain is a Prufer domain and so every nonzero ideal of Dedekind domain is a maximal t-ideal.)

Call an integral domain D a t-local domain if D is quasi local with maximal ideal M a t-ideal, then D is a HoFD in that every nonzero non unit of D is a homogeneous element and hence is uniquely expressible as a product of mutually t-comaximal elements. Now every one dimensional local ring being t-local is a HoFD, in view of this the following proposition provides a valuable contrast.

Proposition 3. *Let D be an integral domain with each nonzero non unit a rigid element. Then D is a valuation domain.*

Proof. Let x, y be two nonzero non units. Then xy being a nonzero non unit, and hence rigid, gives $x|y$ or $y|x$. Thus for every pair of elements we have one dividing the other. □

Finally it appears that, very few restrictions other than the property $*$ will make semirigid domains into GCD domains. For example a Krull domain is atomic, and hence a semirigid domain, but not all Krull domains are UFDs. So some products of rigid elements are not rigid. However the following simple statement holds.

Proposition 4. *An atomic domain D is a UFD if and only if for every pair of atoms a, b, $(a,b)_v \neq D$ implies that ab is rigid.*

Since the proof of the significant part is direct, we leave the proof to the reader.

Finally, note that if h is a homogeneous element of an integral domain, then every non unit factor t of h is in $M(h)$ the unique maximal t-ideal containing h. Thus for every pair of non unit factors u, v of a homogeneous element h we have

$(h,q)_v \neq D$. This leads to the question: Call a nonzero non unit q of an integral domain D a pre-homogeneous element if for every pair r,s of non unit factors of q we have $(r,s)_v \neq D$. Must a pre-homogeneous element be homogeneous?

Generally the answer is no, as every rigid element is pre-homogeneous and a rigid element may not be homogeneous. For example, as we have already mentioned, an atom is rigid and an atom may belong to more than one maximal t-ideals. However in some integral domains a pre-homogeneous element may well be homogeneous.

Proposition 5. *In a domain D with PSP property, every pre-homogeneous element is homogeneous.*

Proof. Let q be a pre-homogeneous element of the PSP domain D. Suppose that q is not homogeneous. Then there are at least two maximal t-ideals M_1, M_2 containing q. Let $m_1 \in M_1 \backslash M_2$ and $m_2 \in M_2 \backslash M$. Then $(m_1, M_2)_t = D$ and $(m_2, M_1)_t = D$. We can write $(m_1, M_2)_t = (m_1, F_2)_t$ where $F_2 \subseteq M_2$ and similarly $(m_2, M_1)_t = (m_2, F_1)_t$ where F_1 is a finitely generated ideal contained in M_1. Set $G = (q, F_1, F_2, m_1, m_2)$. Now $(q, F_1, m_1) \subseteq M_1$, so $(q, F_1, m_1)_v \neq D$. Since D is a PSP domain, $(q, F_1, m_1)_v \neq D$ means that there is a non unit r such that $(q, F_1, m_1) \subseteq rD$. Similarly we can find a nonzero non unit s in D such that $(q, F_2, m_2) \subseteq sD$, because $(q, F_2, m_2) \subseteq M_2$. Thus $(q, F_1, F_2, m_1, m_2) \subseteq (r, s)$. Yet, $(q, F_1, F_2, m_1, m_2)_t = D$. Whence $(r, s)_t = D$ a contradiction to the assumption that q is sub-homogeneous. Since this contradiction arises from the assumption that q is contained in more than one maximal t-ideals the conclusion follows. □

Corollary 3. *A PSP domain whose nonzero non units are expressible as finite products of pre-homogeneous elements is a HoFD.*

Now as we have seen, a rigid element is pre-homogeneous. we have the following result.

Corollary 4. *A semirigid pre-Schreier domain is a HOFD and consequestly a VFD is a HoFD.*

The proof depends upon the fact that a pre-Schreier domain is PSP, as we have already seen. Moreover in, a PSP domain and hence, in a pre-Schreier domain a rigid element is homogeneous. Now use Proposition 1. It would be quite instructive to compare Corollary 4 with Proposition 3.1 of [14]. Now Could this author be wrong? What I have to call pre-homogeneous was once called homogeneous and according to Theorem 2.3 of [5], a completely primal (pre-) homogeneous element is t-pure (modern day homogeneous).

3. Discussion

Finally, let me note that, in the good old days, Dan Anderson wrote a nice chapter [1], highlighting some of my work, solo or joint with him, on unique factorization in domains which may not pass as UFDs. In this paper he also mentions generalized UFDs (GUFDs) and mentions [3] as its source. Actually, the theory of GUFDs was included in the first chapter of my doctoral dissertation. Briefly a GUFD is a semirigid domain where the rigid elements, called prime quanta, have all the properties of prime powers. (A rigid element q is a prime quantum if q is completely primal such that every power of q is rigid and for each non unit factor h of q we have $q|h^n$ for some n.) I showed a GUFD to be a GCD domain that was also a generalized

Krull domain (GKD) that is a domain D such that (a) D is a locally finite intersection of localizations at all height one primes of D and (b) D_P is a valuation domain for each height one prime of D. It so transpired that later, in [4], domains with just the (a) part were studied as weakly Krull domains. And as noted on page 350 of [5], just above Corollary 3.8, a weakly Krull domain that is a GCD domain is a GUFD. A copy of my thesis is available here [21]. With some effort my thesis can be downloaded from here: https://ethos.bl.uk/OrderDetails.do?uin=uk.bl.ethos.704293

One of the reasons for bringing up my thesis and Dan's paper [1] is the sight of, "It is easy to see that if D is not a field, then D is a weakly factorial GCD-domain if and only if D is a weakly Matlis GCD-domain with t-dim$(D) = 1$." in [14]. Why "weakly factorial GCD-domain" and not a GUFD? Next why "weakly Matlis GCD-domain" and not a GCD IRKT or not a semirigid GCD domain? By Theorem 3.8 of [1] they are the same things! Next, why so much emphasis on weakly factorial domains? They only deal with a special case? Finally, I am grateful for the authors mentioning my paper [22], but why add "pre-Schreier domains have some "nice" properties"? (Without qualifying the quotes on "nice"!) Were they trying to poke fun? (Was the referee sleeping? Or is the referee part of the problem?)

Aside from gripes, I mentioned my thesis because, in the definition of a prime quantum there is a novel trick that makes sure that the product of two rigid elements is rigid, esuring the GCD property. But of course that would only get you GUFDs.

References

[1] D.D. Anderson, Non-atomic unique factorization in integral domains, in Properties of Commutative Rings and Modules (ed. S. Chapman) SRC Press Boca Raton, (2005), 1-21.

[2] D.D. Anderson, D.F. Anderson, M. Zafrullah, Factorization in integral domains, J. Pure Appl. Algebra 69, 1–19 (1990).

[3] D.D. Anderson, D.F. Anderson and M. Zafrullah, On generalized unique factorization, Bollettino U. M. I. (7) 9-A (1995), 401-413.

[4] D.D. Anderson, J. Mott and M. Zafrullah, Finite character representations for integral domains, Boll. Un. Mat. Ital. 6 (1992), 613 630.

[5] D. D. Anderson, J. L. Mott and M. Zafrullah, Unique factorization in non-atomic integral domains, Boll. Unione Mat. Ital. Ser. 8 2-B (1999) 341–352.

[6] D.D. Anderson and M. Zafrullah, Independent locally-finite intersections of localizations, Houston J. Math. 25 (1999), 433-452.

[7] D.D. Anderson and M. Zafrullah, On [redacted]-Semi-homogeneous Integral Domains. In: Badawi A., Coykendall J. (eds) Advances in Commutative Algebra. Trends in Mathematics. Birkhäuser, Singapore, (2019).

[8] D.D. Anderson, M. Zafrullah, The Schreier property and Gauss' lemma, Boll. UMI 10 (2007) 43–62.

[9] D. D. Anderson and M. Zafrullah, Splitting sets and weakly Matlis domains, in: Commutative Algebra and its Applications, eds. M. Fontana et al. (Walter de Gruyter, New York, 2009), pp. 1–8.

[10] A. Bouvier, Le groupe des classes d'un anneau integre, 107'eme Congres des Soci'et'es Savantes, Brest fasc. IV (1982), pp. 85–92.

[11] G.W. Chang, Unique factorization property of non-unique factorization domains, J. Algebra Appl. (2020).

[12] P.M. Cohn, Bézout rings and their subrings, Proc. Camb. Philos. Soc. 64 (1968) 251–264.

[13] P.M. Cohn, Unique factorization domains, Amer. Math. Monthly, 80 (1)(1973) 1-18.

[14] G.W. Chang and Andreas Reinhart, Unique factorization property of non-unique factorization domains ii, J. Pure Appl. Algebra 224 (12) (2020), 106430.

[15] J. Elliott, Rings, modules and closure operations, Springer Monographs in Mathematics. Springer, Cham, 2019.

[16] M. Griffin, Some results on v-multiplication rings, Canad. J. Math.19(1967) 710-722.

[17] M. Griffin, Rings of Krull type. J. Reine Angew. Math. 229, 1–27 (1968).
[18] M. Zafrullah, What v-coprimality can do for you, in: J.W. Brewer, et al. (Eds.), Multiplicative Ideal Theory in Commutative Algebra, Springer, 2006, pp. 387–404.
[19] M. Zafrullah, Rigid elements in GCD domains" J. Natur. Sci. and Math. **17** (1977), 7–14.
[20] M. Zafrullah, Semirigid GCD domains, Manuscripta Math. 17(1975), 55-66.
[21] M. Zafrullah, https://lohar.com/researchpdf/Zafrullah_Dissertation.pdf
[22] M. Zafrullah, On a property of pre-Schreier domains, Commun. Algebra 15 (1987) 1895–1920.

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