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SEMIRIGID GCD DOMAINS II

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Dedicated to the memory of Paul Cohn

ABSTRACT. Let D be an integral domain with quotient field K, throughout. Call two elements $x, y \in D\backslash\{0\}$ v-coprime if $xD \cap yD = xyD$. Call a nonzero non unit r of an integral domain D rigid if for all $x, y|r$ we have $x|y$ or $y|x$. Also call D semirigid if every nonzero non unit of D is expressible as a finite product of rigid elements. We show that a semirigid domain D is a GCD domain if and only if D satisfies $*$: product of every pair of non-v-coprime rigid elements is again rigid. Next call $a \in D$ a valuation element if $aV \cap D = aD$ for some valuation ring V with $D \subseteq V \subseteq K$ and call D a VFD if every nonzero non unit of D is a finite product of valuation elements. It turns out that a valuation element is what we call a packed element: a rigid element r all of whose powers are rigid and $\sqrt{rD}$ is a prime ideal. Calling D a semi packed domain (SPD) if every nonzero non unit of D is a finite product of packed elements, we study SPDs and explore situations in which a variant of an SPD is a semirigid GCD domain.

1. Introduction

Let D be an integral domain with quotient field K. Some recent research is a treasure trove of new ideas that are linked to some old ideas in an uncanny fashion. Some of these are concepts such as a homogeneous element of Chang [14], one that belongs to a unique maximal t-ideal and a valuation element of Chang and Reinhart [15], i.e. an element a such that $aV \cap D = aD$ for some valuation ring V with $D \subseteq V \subseteq K$. It is a beautiful notion and its properties are linked with the so called rigid elements that I knew and worked with, long ago. The aim of this note is to highlight the connections, smooth out the kinks caused by changes in terminology and to present some new results.

Chang [14] calls a domain a HoFD if every nonzero non unit of D is expressible as a product of mutually t-comaximal homogeneous elements and says HoFDs were first studied in [7], of course with different terminology. (A homogeneous element was "t-pure" and a HoFD was a semi t-pure domain.) According to [15, Corollary 1.2] a valuation element a of a domain D has the property that for all $x, y|a$ we have $x|y$ or $y|x$. This makes a a rigid element of Cohn [17]. But the valuation elements of [15] beat Cohn's rigid elements by a mile in properties. To be exact, let's call an element r of D rigid if r is a nonzero non-unit such that for all $x, y|r$ we have $x|y$ or $y|x$. Then a valuation element a, in a VFD, is rigid such that every power of a is rigid and $\sqrt{a}$ is a prime [15, Corollaries 1.2, 1.11], here $\sqrt{a}$ represents the radical

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of the ideal (a). Let's call an element with these properties a packed element. They also show that a VFD is Schreier, an integrally closed integral domain D in which every nonzero element x is primal, i.e., has the property that for $y, z \in D\backslash\{0\}$ $x|yz$ implies that $x = rs$ where $r|y$ and $s|z$. A domain all of whose nonzero elements are primal is called a pre-Schreier domain. Let's call D semirigid (resp., semi homogeneous, semi packed) if every nonzero non unit of D is expressible as a finite product of rigid (resp., homogeneous, packed) elements of D. Let's also call a pre-Schreier SPD a t-SPD. The trouble with the semirigid (resp., semi homogeneous) domains is that they are very general. For example every irreducible element, i.e., a nonzero non-unit a such that $a = \alpha\beta \Rightarrow \alpha$ is a unit or β is, is rigid. But the atomic domains, i.e., domains whose nonzero non-units are expressible as finite products of irreducible elements, often have little or no form of uniqueness of factorization [2]. For example, in $D = F[[X^2, X^3]]$, that is Noetherian and hence atomic, the elements X^2 and X^3 are irreducible, and $(X^2)^3 = (X^3)^2 = X^6$. That is X^6 has two distinct factorizations. On the other hand, as we shall show, semi homogeneous domains are actually HoFDs and so do have a sort of uniqueness of factorization, but only just.

One way of getting such wayward concepts to deliver unique factorization of some sort is to bring in a somewhat stronger notion of coprimality and some conditions. Call two elements a, b of a domain D. v-coprime if $aD \cap bD = abD$. Obviously a, b are v-coprime if and only if $(a, b)^{-1} = D$, if and only, if $((a, b)^{-1})^{-1} = (a, b)_v = D$, where $A \mapsto A_v = (A^{-1})^{-1}$ is the usual star operation called the v-operation on $F(D)$, the set of nonzero factional ideals of D. The notion of v-coprimality has been discussed in detail in [29], where it is shown, in somewhat general terms, that if, for $a, b, c \in D\backslash\{0\}$, $(a, b)_v = D$ and $a|bc$, then $a|c$. It was also shown in [29, Proposition 2.2] that for $r_1, ..., r_n, x \in D\backslash\{0\}$ $(r_1...r_n, x)_v = D$ if and only if $(r_i, x)_v = D$. Let's call two homogeneous (resp., rigid) elements a, b similar, denoted $a\tilde{}b$, if $(a, b)_v \neq D$. We plan to show that a semi homogeneous domain is a "HoFD" because the product of every pair of similar homogeneous elements of D is again a homogeneous element, of D, similar to them. We also show in Section 2 that a semirigid domain is a semirigid GCD domain if and only if the product of each pair of non-v-coprime rigid elements is rigid and give examples to show that the product of two non-v-coprime rigid elements may not be rigid. We shall also give examples to show that a homogeneous element may not be rigid and a rigid element may not be homogeneous. On the semi packed front, we establish a part of the theory and then introduce semi-t-packed domains (t-SPDs) as a concept parallel to VFDs in section 3 and give at least one example of a t-SPD that is not a semirigid GCD domain. Here a t-SPD is a SPD in which every packet element is completely primal, a primal element with all factors primal. We also explore the conditions that make a t-SPD (resp a VFD) into a semirigid GCD domain.

It seems best to give the reader an idea of the v- and the t-operations and some related concepts that we shall have the occasion to use. For $I \in F(D)$, the set $I^{-1} = \{x \in K | xI \subseteq D\}$ is again a fractional ideal and thus the relation v: $I \mapsto I_v$ is a function on $F(D)$. This function is called the v-operation on D. Similarly the relation t: $I \mapsto I_t = \cup\{F_v | \ 0 \neq F$ is a finitely generated subideal of $I\}$ is a function on $F(D)$ and is called the t-operation on D. These are examples of the so called star operations. The reader may consult Jesse Elliott's book [20] for these operations. A fractional ideal I is called a v-ideal (resp. a t-ideal) if $I_v = I$ (resp., $I_t = I$).The rather peculiar definition of the t-operation allows one to use Zorn's Lemma to prove that each integral domain that is not a field has at least one integral t-ideal maximal among integral t-ideals. This maximal t-ideal is prime and every proper, integral t-ideal is contained in at least one maximal t-ideal. A minimal prime of a t-ideal is a t-ideal and thus every height one prime is a t-ideal. The set of all maximal t-ideals of a domain D is denoted by $t\text{-}Max(D)$. It can be shown that $D = \cap_{M\in t\text{-}Max(D)} D_M$. A fractional ideal I is said to be t-invertible if $(II^{-1})_t = D$. A domain in which every nonzero finitely generated ideal is t-invertible is called a Prufer v-multiplication domain (PVMD), a Prufer domain is a PVMD with every nonzero ideal a t-ideal. Griffin [21] showed that D is a PVMD if and only if D_M is a valuation domain for each maximal t-ideal M of D. Given any domain D the set $t\text{-}inv(D)$ of all t-invertible fractional t-ideals of D is a group under the t-operation (i.e., $I \times_t J = (IJ)_t$). The group $t\text{-}inv(D)$ has the group $P(D)$ of nonzero principal fractional ideals as its subgroup. The t-class group of D is the quotient group $Cl_t(D) = t\text{-}inv(D)/P(D)$. What makes this group interesting is that if D is a Krull domain $Cl_t(D)$ is the divisor class group of D and if D is a Prufer domain, $Cl_t(D)$ is the ideal class group of D. Of interest for this note is the fact that a PVMD D is a GCD domain if and only if $Cl_t(D)$ is trivial. This group was introduced in [12].

Next, note that if D is pre-Schreier then $Cl_t(D)$ is trivial, [12]. According to Cohn [16], if S is a set multiplicatively generated by completely primal elements of an integrally closed domain D such that D_S is a Schreier domain, then D is a Schreier domain. This Theorem is usually referred to as: Cohn's Nagata type Theorem for Schreier domains. Our terminology is standard or is explained at the point of entry.

2. Semirigid GCD domains

Let's note that for a finitely generated nonzero ideal $I = (x_1, ..., x_n)$ we have $I_v = I_t$, so $x_1, ..., x_n$ being v-coprime (i.e., $(x_1, ..., x_n)_v = D$) is the same as $x_1, ..., x_n$ being t-comaximal (i.e., $(x_1, ..., x_n)_t = D$), which boils down to: $x_1, ..., x_n$ do not share any maximal t-ideal. We also note that a is a homogeneous element if aD is a t-homogeneous ideal in the sense of [9] and, sort of, following the convention of [9] we shall denote by $M(a)$ the maximal t-ideal containing the homogeneous element a. Indeed we have $M(a) = \{x \in D| \ (x, a)_v \neq D\}$ (cf. [9, (2) Proposition 1]). The following two results can be proved using Theorem 3.1 of [7], but due to the change of terminology, mentioned in the introduction, it seems safe to redo them here. (I plan to address the change of terminology later and suggest a way to patch things up.)

Lemma 2.1. *Let a and b be two homogeneous elements of D then $(a, b)_v \neq D$ if and only if (a, b) is contained in the same maximal t-ideal if and only if ab is a homogeneous element.*

Proof. Let b be a homogeneous element belonging to the maximal t-ideal P. For any nonzero finitely generated ideal A, $(A, b)_v \neq D$ implies that that $A \subseteq P$. This is because $(A, b)_v \neq D$ implies (A, b) has to be contained in some maximal t-ideal and P is the only maximal t-ideal that contains b. So $A \subseteq P$. Now $(a, b)_v \neq D$ implies that a, b both belong to the same maximal t-ideal say P. Next note that $x \in M(a) \Leftrightarrow (x, a)_v \neq D$. So $x \in M(a)$ implies x belongs to P. Thus $M(a) = P$ and similarly $M(b) = P$ forcing $M(a) = M(b)$. Suppose ab belongs to a maximal

t-ideal P. Then $a \in P$ or $b \in P$. If $a \in P$, then $M(a) = P$. But as $(b,a)_v \neq D$, $M(a) = M(b)$ whence ab is a homogeneous element, as $P(a) = P(b)$ is the only maximal t-ideal containing ab. Finally if ab is t-homogeneous then, by definition, $(a,b)_v \neq D$. □

Proposition 1. *An integral domain D is a HoFD if and only if D is a semi homogeneous domain.*

Proof. Suppose that D is a semi homogeneous domain. Lemma 2.1 shows that the product of every pair of similar homogeneous elements of D is homogeneous. Let $x = h_1h_2...h_n$ where each of h_i is a homogeneous element. Now $M_1, ...M_r$ be the set of distinct maximal t-ideals containing h. Let $H_j = \Pi h$ where h ranges over $h_i \in M_j$. By Lemma 2.1, H_j are homogeneous and mutually t-comaximal. Thus we have $x = \Pi_{j=1}^{r} H_j$ where H_i are mutually v-coprime homogeneous. The converse is obvious. □

It was shown in [24] that if a nonzero non unit x in a GCD domain is expressible as a finite product of rigid elements, then x is uniquely expressible as a product of finitely many mutually coprime rigid elements. Thus showing that in a semirigid GCD domain every nonzero non unit x is expressible uniquely as a product of mutually coprime elements. So a valuation ring V of any rank is an example of a semirigid GCD domain and so is a polynomial ring over V. Griffin, in [22], called a domain D an Independent Ring of Krull type (IRKT) if D has a family of prime ideals $\{P_\alpha\}_{\alpha \in I}$ such that (a) D_{P_α} is a valuation domain for each $\alpha \in I$, (b) $D = \cap_{\alpha \in I} D_{P_\alpha}$ is locally finite and (c) No pair of distinct members of $\{P_\alpha\}_{\alpha \in I}$ contains a nonzero prime ideal. It was shown in Theorem 5 of [24] that a semirigid GCD domain is indeed an IRKT. Also, it was shown in Theorem B of [25] that a GCD IRKT is a semirigid GCD domain. Later, a domain satisfying only (b) and (c) above, requiring that P_α are maximal t-ideals, was called in [11] a weakly Matlis domain. An IRKT is a PVMD, [22]. Also, suppose that D is a weakly Matlis GCD domain. Noting that a GCD domain is a PVMD which makes localization at each maximal t-ideal a valuation domain we have each of D_{P_α} a valuation domain, in the definition of a weakly Matlis domain and making it an IRKT. Finally, a GCD IRKT is a semirigid GCD domain, by Theorem B of [25]. We now show that introducing a simple property $*$: for every pair r, s of rigid elements $(r,s)_v \neq D \Leftrightarrow rs$ is rigid, we can make a semirigid domain into a semirigid GCD domain.

Lemma 2.2. *Let D be a semirigid domain with $*$: for every pair r, s of rigid elements $(r,s)_v \neq D \Leftrightarrow rs$ is rigid. Then the following hold. (1) Suppose that r, s are two similar rigid elements. Then r and s are comparable, i.e., $r|s$ or $s|r$, (2) If r is a rigid element and s, t are rigid elements, each similar to r, then s and t are similar, (3) A finite product of mutually similar rigid elements is rigid similar to each of the factors and (4) if a rigid element r divides a product $x = x_1x_2...x_n$ of mutually v-coprime rigid elements $x_1, ..., x_n$ then r divides exactly one of the x_i, in a semirigid domain with property $*$.*

Proof. (1). Straightforward, as rs is rigid.

(2). $r|s$ or $s|r$ and $r|t$ or $t|r$. Four cases arise (i) $r|s$ and $r|t \Rightarrow (s,t)_v \neq D$, (ii) $r|s$ and $t|r \Rightarrow t|s$ (iii) $s|r$ and $r|t \Rightarrow s|t$ (iv) $s|r$ and $t|r \Rightarrow s|t$ or $t|s$. In each case we have $s\tilde{\ }t$.

(3). Suppose that D is semirigid with the given property $(*)$. Using induction, one can show that in a semirigid domain with $(*)$, a finite product of mutually similar rigid elements is rigid. This is how it can be accomplished: We know that the product of any two similar rigid elements is rigid. Assume that we have established that the product of any set of n of rigid elements, $r_1, r_2, ..., r_n$, similar to one of them and hence, by (2), similar to each other, is rigid. Let $\boldsymbol{r} = r_1 r_2 ... r_n$ and let s be a rigid element similar to, one and hence, each of r_i and hence to $\mathbf{r}$. But then by $*$, $\mathbf{r}s$ is rigid.

(4). Note that $(r, x)_v = rD \neq D$, because $r|x$. So r cannot be v-coprime to each of x_i, [29, Proposition 2.2]. Now, say, r is non-v-coprime to x_i, x_j for $i \neq j$. But then, by (2), $x_i \tilde{} x_j$ which is impossible because $(x_i, x_j)_v = D$. So r is non-v-coprime to exactly one of x_i, say x_k. Now as D has the property $*$ and as r and x_k are rigid, one of them divides the other. But since $r|x$ already, we conclude that $r|x_k$. □

Proposition 2. *Let D be a semirigid domain. Then every nonzero non unit of D is either rigid or can be written uniquely as a product of finitely many mutually v-coprime rigid elements if and only if $*$: for every pair r, s of rigid elements $(r, s)_v \neq D \Leftrightarrow rs$ is rigid holds.*

Proof. Let $x = r_1 r_2 ... r_n$ be a nonzero non unit of D. Pick r_1 and collect all the rigid factors, from r_i, $(i = 1, ..., n)$, that are similar to r_1. Next suppose that by a relabeling we can write $x = r_1 r_2 ... r_{s_1} r_{s_1+1} ... r_n$ where r_i $(i = 1, ..., s_1)$ are all the rigid factors of x that are similar to r_1. Set $\mathbf{r}_1 = r_1 r_2 ... r_{s_1}$. Note that since, by the procedure, each of r_i $(i = 1, ..., s_1)$ is v-coprime to each of r_{i_1} $(i_1 = s_1 + 1, ..., n)$ we conclude that $\mathbf{r}_1$ is v-coprime to each of r_{i_1} $(i_1 = s_1 + 1, ..., n)$ and, of course, each of r_{i_1} $(i_1 = s_1 + 1, ..., n)$ v-coprime to $\mathbf{r}_1$. Now select all the rigid elements similar to r_{s_1+1} and suppose that by a relabeling we can write $r_{s_1+1} ... r_n = r_{s_1+1} r_{s_1+2} ... r_{s_2} ... r_n$, where r_j $(j = s_1 + 1, ..., s_2)$ are similar to r_{s_1+1}. Set $\mathbf{r}_2 = r_{s_1+1} r_{s_1+2} ... r_{s_2}$. By, Lemma 2.2, $\mathbf{r}_2$ is rigid. Since $r_{s_1+1}, r_{s_1+2}, ..., r_{s_2}$ are each v-coprime to $\mathbf{r}_1$, and so is their product, we conclude that $\mathbf{r}_1$ and $\mathbf{r}_2$ are v-coprime rigid elements. Thus $x = \mathbf{r}_1 \mathbf{r}_2 r_{s_2+1} ... r_n$ and continuing in this manner we can write $x = \mathbf{r}_1 \mathbf{r}_2 ... \mathbf{r}_m$ where $\mathbf{r}_i$ are mutually v-coprime rigid elements.

Now let $x = \mathbf{r}_1 \mathbf{r}_2 ... \mathbf{r}_m$ be a product of mutually v-coprime rigid elements in a domain D with property $*$. Also let $x = \mathbf{s}_1 \mathbf{s}_2 ... \mathbf{s}_n$. We claim that each of the $\mathbf{r}_i$ is an associate of exactly one of the $\mathbf{s}_j$ and hence $m = n$. For this note that by (4) of Lemma 2.2, $\mathbf{r}_1 | \mathbf{s}_1 \mathbf{s}_2 ... \mathbf{s}_n$ implies that $\mathbf{r}_1$ divides exactly one one of the $\mathbf{s}_j$, say $\mathbf{s}_1$, by a relabeling. But then, considering $\mathbf{s}_1 |\ \mathbf{r}_1 \mathbf{r}_2 ... \mathbf{r}$ and noting that $\mathbf{s}_1 \tilde{} \mathbf{r}_1$ we conclude that $\mathbf{s}_1 | \mathbf{r}_1$. This leaves us with $\mathbf{r}_2 ... \mathbf{r}_m = \mathbf{s}_2 ... \mathbf{s}_n$. Now eliminating one by one, in this manner, and noting that $\mathbf{r}_i$, $\mathbf{s}_j$ are non units will take us to the conclusion, eventually.

Conversely let D be a semirigid domain in which every nonzero non unit is either a rigid element or is uniquely expressible as a finite product of mutually v-coprime rigid elements and consider $x = rs$ where r and s are any two similar rigid elements. If in each case rs is rigid, we are done. To ensure that there is no other possibility we proceed as follows. Because x is expressible as a product of finitely many mutually v-coprime rigid elements we can write $rs = r_1 ... r_n$. By (4) of Lemma 2.2, each of r, s divides exactly one of the r_i, so $n \leq 2$. So let $rs = r_1 r_2$, where $r | r_1$ and $s | r_2$. Now this too is impossible because, by assumption, r and s are non-v-coprime while

r_1 and r_2 are v-coprime and obviously a pair of v-coprime elements (such as r_1, r_2) cannot have factors like $r|\acute{r}_1$ and $s|r_2$ with $(r,s)_v \neq D$. (For this note that $r|\acute{r}_1$ and $s|r_2$ implies that $(\acute{r}_1, r_2) \subseteq (r,s)$ forces $D = (\acute{r}_1, r_2)_v \subseteq (r,s)_v \neq D$, which is a contradiction.) □

Theorem 2.3. *A semirigid domain with property $*$ is a GCD domain.*

Proof. Let r be a rigid element and x be any nonzero non unit of D. By Proposition 2 $x = x_1x_2...x_n$ where x_i are mutually v-coprime rigid elements. Two cases arise: (a) r is v-coprime to each of x_i and (b) r is non-v-coprime to one (and exactly one) say x_1 of the x_i. (By Lemma 2.2 r cannot be non-v-coprime to more than one). Now in the presence of property $*$, $r|x_1$ or $x_1|r$. In case (a) $(r,x)_v = D$ and in case (b) if $r|x_1$ then $r|x$ and so $(r,x)_v = rD$. Finally if $x_1|r$, then since $(r,x_i)_v = D$ for $i = 2,...,n$, we must have $(r/x_1, x_i)_v = D$ for $i = 2,...,n$. But that gives $(r/x_1, x_2...x_n)_v = D$, forcing $(r,x)_v = x_1D$. Throwing in the case when x is a unit we conclude that if r is a rigid element and x any nonzero element of a domain with property $*$, then $(r,x)_v = hD$ a principal ideal. We next show that r is primal. For this we let $r|xy$ for some $x,y \in D\backslash\{0\}$. Letting $x = sx_1$ where s is such that $(r,x)_v = sD$ we have $r = st$ where $(t,x_1)_v = D$. Now $st|sx_1y$ leads to $t|x_1y$, which in turn leads to $t|y$ because $(t,x_1)_v = D$. Whence $r = st$ where $s|x$ and $t|y$. Since the choice of x,y was arbitrary, we conclude that every rigid element r of D is primal. Combine the information that products of primal elements are primal ([16]) with the fact that D is semirigid to conclude that D is indeed pre-Schreier. Now in a pre-Schreier domain a rigid element is homogeneous [7, Theorem 2.3] and so belongs to a unique maximal t-ideal $P(r) = \{x \in D| (r,x)_v \neq D\}$. Once we have shown that every rigid element of a semirigid domain D is homogeneous we can conclude that D is a HoFD. Next, in view of proof of (3) of Theorem 3.1 of [7], one can say that if D is a HoFD and M a maximal t-ideal of D and if $0 \neq xD_M \subseteq MD_M$ then $xD_M \cap D = x'D$ where x' is a homogeneous element contained in M such that $xD_M = x'D_M$. So let M be a maximal t-ideal of D and let $0 \neq hD_M, kD_M \subseteq MD_M$. Then we have $hD_M \cap D = h'D$ and $kD_M \cap D = k'D$ where h' and k' are homogenous elements contained in M and hence similar. But in our domain D with property $*$, h', k'similar means $h'D \subseteq k'D$ or $k'D \subseteq h'D$. Extending these to D_M we get $hD_M = h'D_M \subseteq k'D_M = kD_M$ or $kD_M = k'D_M \subseteq h'D_M = hD_M$. Thus every pair of principal ideals hD_M, kD_M of D_M is comparable and D_M is a valuation domain. Now as the choice of M was arbitrary we conclude that D_M is a valuation domain for every maximal t-ideal of D and D is a PVMD. But a pre-Schreier PVMD is a GCD domain [13, Corollary 1.5]. □

Recall that a polynomial $f(X) = \sum_{i=0}^{i=n} a_iX^i$ is primitive if the coefficients a_i of f have no non unit common factor and f is super primitive if the coefficients $a_0, ..., a_n$ are v-coprime. Also a domain D has the PSP property if every primitive polynomial is super primitive.

Corollary 1. *For an integral domain D the following are equivalent.*

(1) D is a semirigid domain with property $*$,
(2) D is a semirigid GCD domain,
(3) D is a semi homogeneous GCD domain,
(4) D is a HoFD PVMD,

(5) D is a weakly Matlis GCD domain,

(6) D is a GCD VFD,

(7) D is a UVFD,

(8) D is a VFD such that product of every pair of non-coprime valuation elements is again a valuation element,

(9) D is a pre-Schreier semirigid domain such that products of pairs of non-corime rigid elements are again rigid and

(10) D is a semirigid PSP domain such that every element of D is rigid or is expressible as a product of mutually v-coprime rigid elements.

Proof. (1) $\Rightarrow$ (2). This follows from Theorem 2.3.

(2) $\Rightarrow$ (3). Let r be a rigid element of a GCD domain D and consider $P(r) = \{x \in D | (x, r)_v \neq D\}$. By [24, Lemma 1] $P(r)$ is a prime ideal. To see that $P(r)$ is a t-ideal note that because D is a GCD domain, $(x, r)_v \neq D \Leftrightarrow x = a\rho$ where ρ is a non unit factor of r. Thus $x_1, x_2, ..., x_n \in P(r) \Rightarrow (x_1, x_2, ..., x_n) \subseteq (\rho_1)$ where ρ_1 is a non unit factor of r. But then, $x_1, x_2, ..., x_n \in P(r) \Rightarrow (x_1, x_2, ..., x_n)_v \subseteq P(r)$. Thus $P(r)$ is a t-ideal. Finally, let M be a prime ideal properly containing $P(r)$ and let $y \in M \backslash P(r)$. By the definition of $P(r)$, $(y, r)_v = D$. But then $M_t = D$ and this shows that $P(r)$ is actually a maximal t-ideal. Finally, using the definition, it can be easily established that for any pair of rigid elements r, s of D, $P(r) = P(s)$ if and only if $r\tilde{\ }s$. Thus every rigid element, in a GCD domain D, belongs to a unique maximal t-ideal and hence is a homogeneous element. Consequently a semirigid GCD domain is a semi homogeneous GCD domain.

(3) $\Rightarrow$ (4). Note that a semi homogeneous domain is a HoFD, by Proposition 1 and, it is well known that, a GCD domain is a PVMD.

(4) $\Rightarrow$ (5). Since a HoFD D is a weakly Matlis domain with $Cl_t(D) = 0$ [14, Theorem 2.2] or [7, Theorem 3.4] and since a PVMD D with $Cl_t(D) = 0$ is a GCD domain [12, Proposition 2], we conclude that a PVMD HoFD is a weakly Matlis GCD domain.

(5) $\Rightarrow$ (6). Note that Corollary 4.5 of [15] says that a UVFD is a weakly Matlis GCD domain and as a UVFD is a VFD, in particular, we have the conclusion.

(6) $\Rightarrow$ (7). This follows because, according to Theorem 4.2 of [15], a PVMD VFD is a UVFD and so a GCD VFD is a UVFD.

(7) $\Rightarrow$ (8). Let D be a UVFD then D is a VFD. Take two non-coprime valuation elements u, v and using the fact that uv is an element of a UVFD write $uv = a_1a_2...a_n$ where a_i are mutually incomparable and hence mutually coprime. Now $u|a_1a_2...a_n$ implies that $u = u_1u_2...u_n$ where $u_i|a_i$, because a VFD is Schreier [15], see also Cohn [16]. We claim that exactly one of the u_i is non unit. For if say u_1 and u_2 are non units then being factors of coprime elements u_1 and u_2 are incomparable and this contradicts the fact that u is a valuation element (cf. [15, (2) of Corollary 1.2]). So u divides exactly one of the a_i. Similarly v divides exactly one of the a_i. Next u and v cannot divide two distinct a_i for that would make u, v coprime, which they are not. Now suppose that $u|a_1$. Then $v = (a_1/u)a_2...a_n$ and v cannot divide any of $a_2, ..., a_n$ as that would make v coprime with u. So, v must divide (a_1/u). But then $1 = (a_1/uv)a_2...a_n$, forcing $uv = a_1$ and forcing the conclusion that in the VFD D the product of any pair of non coprime valuation elements is again a valuation element.

(8) $\Rightarrow$ (9). Note that as each valuation element is rigid, a VFD is semirigid. Since a VFD is Schreier we can say that D is a pre-Schreier semirigid domain.

Also the product of two non coprime valuation elements being a valuation element translates to the product of two non-v-coprime rigid elements is rigid and that is the property $*$.

(9) $\Rightarrow$ (1). Follows directly, as in a pre-Schreier domain non-coprime is the same as non-$v$coprime.

(2) $\Rightarrow$ (10). This follows directly.

(10) $\Rightarrow$ (1). Let $x = rs$ where r, s are non-v-coprime rigid elements. If rs is not rigid, then $rs = t_1...t_n$ where t_i are mutually v-coprime rigid elements. Now r cannot be non-v-coprime to more than one of the t_i. For if it were it would have non unit factors common with v-coprime elements, forcing r to be non rigid. Thus r divides exactly one of the t_i. Again r and s cannot divide two distinct t_i because then they would be v-coprime. This forces rs to divide exactly one of the t_i, leading to the conclusion that D is a semirigid domain in which the product of every pair of non-v-coprime rigid elements is rigid. □

While Corollary 1 establishes that the older concept of semirigid GCD domains of [24] is precisely the most recent concept of UVFDs of [15], it raises the following question.

Question Must a Schreier semirigid domain be a semirigid GCD domain?

This question becomes interesting in view of the fact that the authors of [15] ask a similar question: is a VFD a semirigid GCD domain? (Actually they ask: Is a VFD a weakly Matlis GCD domain?) The other point of interest is that, according to [16] an atomic Schreier domain is a UFD. In fact, once we recall necessary terminology, we have the following more general result.

Proposition 3. *(cf.,* [10]*, Proposition 3.2). In an integral domain with PSP property, every atom is a prime. Consequently an atomic domain with PSP property is a UFD.*

Here, as we have already noted, a polynomial $f(X) = \sum_{i=0}^{i=n} a_i X^i$ is primitive if the coefficients a_i of f have no non unit common factor and f is super primitive if the coefficients $a_0, ..., a_n$ are v-coprime. Also a domain D has the PSP property if every primitive polynomial is super primitive. Now as was indicated in [10] a domain with PSP property is much more general than a pre-Schreier domain. Lest hopes run too high, we hasten to offer the following example of a semirigid Schreier domain in which the product of two rigid elements is not rigid.

Example 2.4. Let $\mathbb{Z}$ denote the ring of integers, let $\mathbb{Q}$ be the ring of rational numbers and let X, Y be two indeterminates over $\mathbb{Q}$. Construct the two dimensional regular local ring $R = \mathbb{Q}[[X, Y]]$ and for p a prime element set $D = \mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[[X, Y]]$. This ring D is a semirigid Schreier domain with two rigid elements X, Y such that $(X, Y)_v \neq D$, yet XY is not rigid.

Illustration: Indeed D is a quasi local ring with maximal ideal principal and of course X and Y are divisible by every power of p. That D is integrally closed follows from the fact that $D \subseteq \mathbb{Q}[[X, Y]]$ which is integrally closed, that $\mathbb{Z}_{(p)}$ is integrally closed and that X and Y are divisible by powers of p. For D being Schreier let S be multiplicatively generated by p. Then S is generated by completely primal elements and D_S is a UFD. Hence D is Schreier, by Cohn's Nagata type Theorem. Now look at X. Every factor of X of the form p^r or X/p^s. So any pair of factors of X is one of the forms: $(p^r, p^s), (p^r, X/p^s), (X/p^r, X/p^s)$, $r, s \geq 0$, and in each case one of them divides the other. Same with Y. Now $(X,Y)_v \neq D$, because $p|X,Y$. So X and Y are non-v-coprime rigid elements of D. Yet XY cannot be rigid, because X does not divide Y. Finally, as $\mathbb{Q}[[X,Y]]$ is a UFD with each prime an element $f(X,Y)$ such that $f(0,0)=0$ we conclude that for each prime f of $\mathbb{Q}[[X,Y]]$, f/p^r is rigid in $\mathbb{Z}_{(p)}+(X,Y)\mathbb{Q}[[X,Y]]$. Since a typical nonzero non-unit of a typical nonzero non unit element of $\mathbb{Q}[[X,Y]]$ is expressible as a finite product of primes $f_1(X,Y)...f_n(X,Y)$ of the ring $D=\mathbb{Z}_{(p)}+(X,Y)\mathbb{Q}[[X,Y]]$ is expressible as a finite product of rigid elements $p^r f_1(X,Y)/p^{r_1}...f_n(X,Y)/p^{r_n}$.

Remark 2.5. The above example has often appeared, in various capacities, in papers in which Dan Anderson and I have been coauthors, see e.g. [7, page 344], [11], though the current application is much more elaborate and hence different. Also, as we shall see in the next section that, the above example can have another interpretation/application.

It was shown in the proof of (2) $\Rightarrow$ (3) of Corollary 1 that a rigid element is a homogeneous element. However a rigid element may not generally be a homogeneous element. For an atom is rigid but, say, in a Krull domain with torsion divisor class group an atom can be in more than one height one prime ideals which can be shown to be maximal t-ideals. (For a concrete example $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain in which 3 is well known to be an irreducible element, but $(3)=(3,1-2\sqrt{-5})(3,1+2\sqrt{-5})$ where $(3,1-2\sqrt{-5}),((3,1+2\sqrt{-5})$ are height one prime ideals and hence maximal (t-) ideals of $\mathbb{Z}[\sqrt{-5}]$. (Recall here that a Dedekind domain is a Prufer domain and so every nonzero ideal of a Dedekind domain is a t-ideal.)

Call an integral domain D a t-local domain if D is quasi local with maximal ideal M a t-ideal, then D is a HoFD in that every nonzero non unit of D is a homogeneous element and hence is uniquely expressible as a product of mutually t-comaximal elements. Now every one dimensional local ring being t-local is a HoFD and in view of this fact the following proposition provides a valuable contrast.

Proposition 4. *Let D be an integral domain with each nonzero non unit a rigid element. Then D is a valuation domain.*

Proof. Let x,y be two nonzero non units. Then xy being a nonzero non unit, and hence rigid, gives $x|y$ or $y|x$. Thus for every pair of elements we have one dividing the other. □

It appears that, very few restrictions other than the property $*$ will make semirigid domains into GCD domains. For example a Krull domain is atomic, and hence a semirigid domain, but not all Krull domains are UFDs. So some products of rigid elements are not rigid. However the following statement holds.

Proposition 5. *An atomic domain D is a UFD if and only if for every pair of atoms a,b, $(a,b)_v \neq D$ implies that ab is rigid.*

Proof. We show that if D is atomic such that $(a,b)_v \neq D$ implies that ab is rigid for every pair of atoms a,b, then every atom is a prime. For this take an atom a and some other atom x. Note that if $(a,x)_v \neq D$ then a and x are associates because, by the condition, ax is rigid and so $a|x$ or $x|a$. If $a|x$, then $x=ar$ and because a is a non unit and x being an atom r must be a unit, by the definition of an atom. Similarly if $x|a$, we conclude that x and a are associates. Taking the contrapositive we conclude that in an atomic domain with the given property, if two atoms are non-associates then they are v-coprime. Now let a be an atom and let x be any element in our domain. We claim that in D, $a \nmid x$ implies that $(a,x)_v = D$. For if $x = x_1...x_n$ is any atomic factorization of x and $(a,x)_v \neq D$ then $(a,x_i)_v \neq D$ for some i. (Else if x_i are all v-coprime to a, then so is their product.)

Finally let r,s be any two nonzero elements of D and let $a|rs$. Then $a|r$ or $a|s$. For if $a \nmid r$ and $a \nmid s$, then $(a,r)_v = D$ and $(a,s)_v = D$ which forces $(a,rs)_v = D$. So the atom that we picked is a prime. Whence every atom of D is a prime. But an atomic domain in which every atom is a prime must be a UFD. The converse is obvious. □

Finally, note that if h is a homogeneous element of an integral domain, then every non unit factor t of h is in $M(h)$ the unique maximal t-ideal containing h. Thus for each pair of non unit factors u,v of a homogeneous element h we have $(h,q)_v \neq D$. This leads to the question: Call a nonzero non unit q of an integral domain D a pre-homogeneous element if for every pair r,s of non unit factors of q we have $(r,s)_v \neq D$. Must a pre-homogeneous element be homogeneous?

Generally the answer is no, as every rigid element is pre-homogeneous and a rigid element may not be homogeneous. For example, as we have already mentioned, an atom is rigid and an atom may belong to more than one maximal t-ideals. However in some integral domains a pre-homogeneous element may well be homogeneous. The reason for introducing this new terminology, here, is that the term "homogeneous" was used for an element, in [7], in the set up of pre-Schreier domains saying that h is homogeneous if h is a non unit such that for all non-units $a,b|h$ we have $(a,b)_v \neq D$. At that time there was no clear concept of homogeneous ideals. This changed in [18] and we called a proper (nonzero) $*$-finite $*$-ideal A of D homogeneous if A is contained in a unique maximal $*$-ideal. Now Chang [14] calls an element h a homogeneous element if hD is a (t-) homogeneous ideal and, as we have shown above, this new homogeneous isn't the old homogeneous. There are two ways of dealing with the situation. Grin and bear it or smooth things over, by introducing some new terminology. I have decided to take the latter approach.

Proposition 6. *In a domain D with PSP property, every pre-homogeneous element is homogeneous.*

Proof. Let q be a pre-homogeneous element of the PSP domain D. Suppose that q is not homogeneous. Then there are at least two maximal t-ideals M_1, M_2 containing q. Let $m_1 \in M_1 \backslash M_2$ and $m_2 \in M_2 \backslash M_1$. Then $(m_1, M_2)_t = D$ and $(m_2, M_1)_t = D$. We can write $(m_1, M_2)_t = (m_1, F_2)_t$ where $F_2 \subseteq M_2$ and similarly $(m_2, M_1)_t = (m_2, F_1)_t$ where F_1 is a finitely generated ideal contained in M_1. Set $G = (q, F_1, F_2, m_1, m_2)$. Now $(q, F_1, m_1) \subseteq M_1$, so $(q, F_1, m_1)_t \neq D$ and as (q, F_1, m_1) is finitely generated $(q, F_1, m_1)_t = (q, F_1, m_1)_v$. Since D is a PSP domain, $(q, F_1, m_1)_v \neq D$ means that there is a non unit r such that $(q, F_1, m_1) \subseteq rD$. Similarly we can find a nonzero non unit s in D such that $(q, F_2, m_2) \subseteq sD$, because $(q, F_2, m_2) \subseteq M_2$. Thus $(q, F_1, F_2, m_1, m_2) \subseteq (r,s)$. Yet, $(q, F_1, F_2, m_1, m_2)_t = D$. Whence $(r,s)_t = D$ a contradiction to the assumption that q is pre-homogeneous. Since this contradiction arises from the assumption that q is contained in more than one maximal t-ideals the conclusion follows. □

Corollary 2. *A PSP domain whose nonzero non units are expressible as finite products of pre-homogeneous elements is a HoFD.*

Now as we have seen, a rigid element is pre-homogeneous. we have the following result.

Corollary 3. *A semirigid pre-Schreier domain is a HoFD and consequently a VFD is a HoFD.*

The proof depends upon the fact that a pre-Schreier domain is PSP, as we have already seen. Moreover in, a PSP domain and hence, in a pre-Schreier domain a rigid element is homogeneous. Now use Proposition 1. Note here that what I have to call pre-homogeneous was once called homogeneous and according to Theorem 2.3 of [7], a completely primal pre-homogeneous element is homogeneous.

Yet having established that a VFD is actually a HoFD does not change much. For example we do have an example, in Example 2.4, of a semirigid pre-Schreier domain that is also a HoFD, being a t-local domain, and being a HoFD has no extra effect on it.

3. Semi packed domains

Consider the following conditions satisfied by a nonzero non unit q of D:

(a) Call q power rigid if q is a rigid element such that every positive integral power of q is rigid,

(b) Call q tenacious if q is a completely primal power rigid element.

(d) Call q prime quantum if q is a tenacious element such that for every non unit factor h of q there is a natural number n with $q|h^n$.

(e) Call q a packed element if q is a power rigid element and a packet i.e.$\sqrt{(q)}$ is a prime ideal.

(f) Call q a t-packed element if q is a tenacious packed element.

Call a domain semi tenacious if every nonzero non-unit of D is expressible as a finite product of tenacious elements and call D semi packed (resp., semi t-packed) if every nonzero non unit of D is expressible as a finite product of packed (t-packed) elements. Let's adopt the convention of using $\sqrt{q}$ to indicate $\sqrt{(q)}$ and of calling a minimal prime of (x) a minimal prime of x. As before, we will call x, y comparable if $x|y$ or $y|x$. With this preparation we begin work on developing the theory of SPDs.

Let's start with the observation that if q is a packed element and r a non unit factor of q, then r is at least a power rigid element.

Lemma 3.1. *Let q be a packed element and let r be a nonzero non unit such that $\sqrt{q} \subseteq \sqrt{r}$. Then r and q are comparable, i.e., $r|q$ or $q|r$. Consequently, two packed elements q and r of a domain D are comparable if and only if $\sqrt{q}$ and $\sqrt{r}$ are comparable.*

Proof. Note that $\sqrt{q} \subseteq \sqrt{r}$ implies that $r|q^n$ for some n. But q^n is rigid. Whence the comparability ($r|q$ or $q|r$). The consequently part is obvious. □

Lemma 3.2. *Suppose that a finite product $p_1p_2...p_n$ of packed elements $p_1, ..., p_n$, in a domain D, is such that $\sqrt{p_1p_2...p_n}$ is a prime. Then $p_1p_2...p_n$ is packed and p_i are mutually comparable and so are their products mutually comparable. Consequently if any pair of p_i is incomparable then $\sqrt{p_1p_2...p_n}$ is not a prime.*

Proof. Note that $\sqrt{p_1p_2...p_n} = \cap\sqrt{p_i} = \sqrt{p_j}$ for some j, because $\sqrt{p_1p_2...p_n}$ is a prime. Next as $\sqrt{p_j} \subseteq \sqrt{p_i}$ for each i we have $p_i|p_j^{n_i}$ for some n_i. Whence every power of $p_1...p_n$, divides a power of p_j, forcing $p_1...p_n$ and to be power rigid. That p_i are mutually comparable follows from the fact that $p_1...p_n$ is power rigid. That the products of p_i are mutually comparable follows from the fact that $p_1...p_n$ is power rigid. The consequently part is obvious. □

Lemma 3.3. *Let a, b be two packed elements of a domain D. If a, b are comparable then ab is a packed element.*

Proof. Since a, b are comparable $ab|a^2$ or $ab|b^2$. Thus ab is power rigid. Also if $(a) \subseteq (b)$, then $\sqrt{a} \subseteq \sqrt{b}$ and so $\sqrt{ab} = \sqrt{a} \cap \sqrt{b} = \sqrt{a}$ (Same for $(b) \subseteq (a)$.) □

Observe that if a and b are packed elements such that $a|b$, then a divides some power of b. So if a divides no power of b, i.e. $\sqrt{b} \nsubseteq \sqrt{a}$ then $a \nmid b$. Thus a and b are incomparable if and only if $\sqrt{a}$ and $\sqrt{b}$ are incomparable. These observations lead to the following conclusion.

Proposition 7. *Let D be an SPD. Then the following hold. (1). Every nonzero non unit x of D is expressible as a product of mutually incomparable packed elements, (2) If $x = x_1...x_n$ is a product of mutually incomparable packed elements then the number of minimal primes of x is precisely n.*

Proof. (1). Let $a = p_1p_2...p_r$ be a product of r packed elements. Pick, say, p_1 and a factor say, by some relabeling, p_2 comparable with p_1. Then p_1p_2 is a packed element. Next pick say p_3, by relabeling, such that p_1p_2 and p_3 are comparable. So $(p_1p_2p_3)$ is a packed element. Continuing in this manner we reach $(p_1p_2...p_{n_1})p_{n_1+1}...p_r$ where none of the $p_{n_1+1}...p_r$ is comparable with $(p_1p_2...p_{n_1})$. Repeat the process with $p_{n_1+1}...p_r$, starting with p_{n_1+1} to get $a = (p_1p_2...p_{n_1})(p_{n_1+1}...p_{n_2})$ $p_{n_2+1}...p_r$. If $s_2 = (p_{n_1+1}...p_{n_2})$ is comparable with $s_1 = (p_1p_2...p_{n_1})$, then each of the factors of $(p_{n_1+1}...p_{n_2})$ would be comparable with s_1 by Lemma 3.2. Thus $a = s_1s_2p_{n_2+1}...p_r$. Continuing in this manner we get $a = s_1s_2...s_t$, where s_i are mutually incomparable.

(2). Let $x = x_1...x_n$ be a product of mutually incomparable packed elements. Taking the radical of both sides we get $\sqrt{x} = \sqrt{x_1...x_n} = \sqrt{x_1} \cap \sqrt{x_2} \cap ... \cap \sqrt{x_n} \supseteq \sqrt{x_1}\sqrt{x_2}...\sqrt{x_n}$. Now any minimal prime ideal P of x contains x and hence $\sqrt{x_1}\sqrt{x_2}...\sqrt{x_n}$, where each of $\sqrt{x_i}$ is a prime containing x. Being minimal, P cannot properly contain all prime ideals containing x. Whence $P = \sqrt{x_i}$ for some i. Also as P has to contain one of the $\sqrt{x_i}$ we conclude that the number of minimal primes of x is precisely n. □

Question: Is an SPD D pre-Schreier? As the authors of [15] use only the properties related to the valuation elements being packed elements to establish that a VFD is a Schreier domain, it is fair to conmjecture that an SPD is pre-Schreier. But a theory parallel to VFDs can be developed by taking our packed elements to be tenacious. So we adopt the easier approach and leave our conjecture to an interested reader.

Proposition 8. *A semi-t-packed domain D is pre-Schreier.*

Proof. Indeed in a semi-t-packed domain every packed element is tenacious and so is completely primal we conclude that every element of a semi-tenacious domain is primal and this makes a semi-t-packed domain pre-Schreier. □

Having indicated that a t-SPD is closely related to VFDs it's pertinent to look for other similarities.

Proposition 9. *A semi tenacious domain is a HoFD.*

Proof. Now as a semi tenacious domain D is pre-Schreier we conclude using Proposition 6 that every rigid element and hence every power rigid element of D is homogeneous and consequently D is a HoFD. □

As already mentioned, an element p in a domain D is called a packet if (p) has a unique minimal prime. That is $\sqrt{p}$ is a prime ideal. This notion was introduced in [23] while studying GCD rings of Krull type. Obviously every nonzero non unit of a ring of Krull type has finitely many minimal primes. It turned out that in a GCD domain D a principal ideal xD has finitely many minimal primes if and only if x can be written as a finite product of packets. So a GCD domain was called a Unique Representation Domain (URD), in [23] if for each nonzero non unit x of D the ideal xD had finitely many minimal primes. GCD URDs were presented in [26]. A more general study of URDs was carried out in [19] where a general integral domain was called a URD, via [19, Corollary 2.12], if for every nonzero non unit x of D we can (uniquely) write $xD = (X_1X_2...X_n)_t$, where X_i are mutually t-comaximal t-invertible t-ideals with $\sqrt{X_i}$ prime. Call D a t-treed domain if the set of prime t-ideals of D is a tree under inclusion. Also, using [19, Corollary 2.12], we concluded that a URD is t-treed.

Lemma 3.4. *Let x be a nonzero non unit in a t-SPD. Then x is a packed element if and only if x is a packet.*

Proof. A packed element is a packet anyway. Conversely let x be a packet and suppose that x is a not a packed element. Since D is semi-t-packed we must have $x = p_1p_2...p_r$ where p_i are mutually incomparable packed elements of D. But this is impossible unless $r = 1$, because x has a unique minimal prime. □

Recall that the notion of HoFD was called a semi t-pure domain in [7] not too long ago and, as already noted, that in view of proof of (3) of Theorem 3.1 of [7] one can say that if D is a HoFD and if $0 \neq xD_M \subseteq MD_M$ then $xD_M \cap D = x'D$ where x' is a homogeneous element contained in M such that $x'D_M = xD_M$.

Lemma 3.5. *Let D be a HoFD. Then the following hold: (1) every homogeneous element of D is a packet if and only if D is t-treed, (2) if every homogeneous element of D is a packed element then D is a PVMD and hence a GCD domain.*

Proof. (1) By the proof of Proposition 1, every nonzero non unit x of a HoFD can be written as $x = h_1h_2...h_r$ where h_i are mutually v-coprime (t-comaximal) homogeneous elements. If each homogeneous element is a packet then [19, Corollary 2.12] applies, so D is a URD and hence t-treed. Conversely if D is t-treed and a HoFD then every homogeneous element of D has a unique minimal prime. Next, for (2), note that in a HoFD every nonzero non unit x can be written as $x = x_1x_2...x_n$ where x_i are mutually t-comaximal homogeneous. Since each homogeneous element is packed and hence a packet, D is t-treed by (1). Note that D is a HoFD to start with. Thus if M is a maximal t-ideal of D and $0 \neq mD_M \subseteq MD_M$, then $mD_M \cap D = m'D$ where m' is a homogeneous element of D in M such that $m'D_M = mD_M$. Now let $0 \neq hD_M, kD_M \subseteq MD_M$. Then correspondingly h', k' are homogeneous elements belonging to M. Because D is t-treed and because homogeneous elements are packed we conclude that $\sqrt{h'}$and$\sqrt{k'}$ are comparable. But then so, by Lemma 3.1, are h' and k' comparable in D and consequently $h'D_M, k'D_M$ in D_M. Since $h'D_M = hD_M$ and $k'D_M = kD_M$, we have $hD_M \subseteq kD_M$ or $kD_M \subseteq hD_M$, forcing D_M to be a valuation domain. Now as M is a typical maximal t-ideal, D_M is a valuation domain for each maximal t-ideal of D. Thus D is a PVMD. Again as D is a HoFD, $Cl_t(D) = 0$ and a PVMD D with $Cl_t(D) = 0$ is a GCD domain, as noted in the introduction. □

Theorem 3.6. *TFAE for a t-treed domain D. (1) D is a VFD, (2) D is a t-SPD, (3) D is a semirigid GCD domain, (4) D[X] is a VFD and (5) D[X] is semi t-packed.*

Proof. (1) ⇒ (2). This follows because a VFD is semi-t-packed.

(2) ⇒ (3). Note that a t-SPD is a HoFD and that in a HoFD every nonzero non unit x can be written as $x = x_1x_2...x_n$ where x_i are mutually t-comaximal homogeneous. Now as D is a HoFD being semi-t-packed and treed all homogenous elements are packets, by (1) of Lemma 3.5. Also since D is semi-t-packed, every packet is a packed element by Lemma 3.4. Whence by (2) of Lemma 3.5 D is a GCD domain. Now as a HoFD GCD domain is a GCD weakly Matlis domain which is a semirigid GCD domain.

(3) ⇒ (4). This follows because if D is a semirigid GCD domain then so is $D[X]$ [24] and as indicated in the previous section a semirigid GCD domain is a VFD.

(4) ⇒ (5). This is direct because a VFD is semi-t-packed, according to [**?**].

(5) ⇒ (1). Since $D[X]$ being semi-t-packed entails $D[X]$ being pre-Schreier and $D[X]$ being pre-Schreier requires D to be integrally closed, by Corollary 10 of [5]. Thus D is Schreier. Also $D[X]$ being of finite t-character forces D to be of finite t-character, [4, Corollary 3.3]. Again $D[X]$ being semi-t-packed means ($D[X]$ is a HoFD and so) no two maximal t-ideals of $D[X]$ contain a nonzero prime ideal. But maximal t-ideals of $D[X]$ are either uppers to zero or of the form $P[X]$ where P is a maximal t-ideal. This leads to the conclusion that between any two maximal t-ideals P, Q of D there is no nonzero prime ideal. Combining this piece of information with D being of finite t-character we conclude that D is a weakly Matlis domain. Noting also that $Cl_t(D) = 0$ because D is Schreier, we conclude that D is a HoFD.

Moreover D being t-treed and of finite t-character forces D to be a URD with every nonzero non unit element a product of mutually t-comaximal packets [19, Corollary 2.12]. Now, all we need show is that each packet of D is a packed element. For this let p be a packet of D. Indeed p is a packet in $D[X]$ and hence a packed element by Lemma 3.4, because $D[X]$ is semi-t-packed. But then p is a packed element of D. Thus D is Schreier semi packed (i.e semi-t-packed). Next using the steps taken in the proof of (2) ⇒ (3) we can show that D is a GCD domain. Now D is a GCD domain and a GCD HoFD is a weakly Matlis GCD domain and hence a t-treed VFD. □

The above theorem shows just how close semi-t-packed domains are to VFDs, without being integrally closed. t-SPDs have at least one advantage over VFDs, we have at least one example of a genuine semi-t-packed domain that is not a semirigid GCD domain.

Example 3.7. The ring $D = \mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[[X, Y]]$ in Example 2.4 is precisely an example of a t-SPD.

Illustration: An element of D is either ϵp^{α}, where ϵ is a unit of D and α a non-negative integer or of the form f/p^{β} where f is a non unit of $\mathbb{Q}[[X,Y]]$ and β an integer. As in Example 2.4, it is easy to check that if f is a prime of $\mathbb{Q}[[X,Y]]$, pairs of factors of $(f/p^{\beta})^n$ are comparable for each positive integer n. (This can be done by mimicking the steps taken for $(X/p^{\beta})^n$. Coupling this piece of information with the fact that for f a prime in $\mathbb{Q}[[X,Y]]$, its extension $(f/p^{\beta})(\mathbb{Q}[[X,Y]])$ to $\mathbb{Q}[[X,Y]]$ is a height one prime we conclude that (f/p^{β}) has a unique minimal prime and so is a packed element. Finally as every nonzero non unit of $\mathbb{Q}[[X,Y]]$ is of the form $(f_1)^{n_1}...(f_r)^{n_r}$, where f_i are primes and n_i non-negative integers, we conclude that every nonzero non unit of D is of the form $\epsilon p^{\alpha}(f_1/p^{\beta_1})^{n_1}...(f_r/p^{\beta_1})^{n_r}$, $\alpha, \beta_i, n_i \in \mathbb{Z}, \alpha, n_i \geq 0$ and hence a product of packed elements. Of course we already know that D is Schreier.

The above example establishes the existence of a t-SPD that is not a semirigid GCD domain. (Or actually the existence of a HoFD in which not all homogeneous elements are packed elements.) On the other hand, being integrally closed, it raises the question: Is the ring in Example 3.7 a VFD? Then there is the question: When is a VFD (resp., semi-t-packed domain, semi tenacious domain) a semirigid GCD domain? We list in propositions below some of the "off the cuff" answers to the question: When is a semirigid domain a semirigid GCD domain. Some of these "answers" may address the specific questions about VFDs etc.

Theorem 3.8. *TFAE for an integral domain D. (1) D is a semirigid GCD domain, (2) D is a GCD weakly Matlis domain, (3) D is a PVMD weakly Matlis domain with $Cl_t(D) = 0$, (4) D is an independent ring of Krull type with $Cl_t(D) = 0$, (5) D is an SPD in which products of pairs of non v-coprime packed elements are packed elements, (6) D is a t-SPD whose homogeneous elements are all packed elements, (7) D is a semi t-packed URD, (8) D is a VFD URD, (9) D is a pre-Schreier semirigid domain that is also a URD in which r is rigid $\Leftrightarrow$ r is a packet, (10) D is a PSP semirigid domain in which products of pairs of non coprime rigid elements are again rigid, (11) D is a t-SPD such that for every pair x, y of packed elements we have $(x,y)_v = D$ or one or both of the following hold $(x,y)_v = xD$ or $(x,y)_v = yD$, (12) D is a semirigid domain and for every rigid element r the following holds: For all $x \in D\backslash\{0\}$, $(r,x)_v = hD$ where h is a factor of r and (13) D is a VFD such that every valuation element of D has a GCD with every nonzero element of D.*

Proof. (1) $\Leftrightarrow$ (2). This is (1) $\Leftrightarrow$ (5) of Corollary 1.

(2) $\Leftrightarrow$ (3). This is obvious because a PVMD D with $Cl_t(D) = 0$ is a GCD domain.

(1) $\Leftrightarrow$ (4). This follows from Corollary 3.8 of [7].

(1) $\Leftrightarrow$ (5). It follows from Corollary 1 in that an SPD is a semirigid domain.

(5)$\Rightarrow$ (6). This follows from the fact that two homogeneous elements belonging to the same maximal t-ideal are non-v-coprime and packed elements in a t-SPD are homogeneous.

(6) $\Rightarrow$ (2). This follows from using the fact that a t-SPD is a HoFD and using (2) Lemma 3.5. Now a GCD HoFD is a GCD weakly Matlis domain.

(1) $\Rightarrow$ (7). For this note that a semirigid GCD domain is t-treed and of finite character and so must be a URD, also being a GCD domain every rigid element in D is power rigid and as we have shown that D is a URD, every rigid element in D is a packed element.

(7)$\Rightarrow$ (1). This follows from taking the following steps: (i) noting that a t-SPD is a HoFD and that in a t-SPD a packet is a packed element (Lemma 3.4) (ii) using (2) of Lemma 3.5 and (iii) noting that a GCD HoFD is a "semi t-pure" GCD domain and hence a semirigid GCD domain by Corollary 3.8 of [7].

(1) $\Rightarrow$ (8). This follows from the fact that every rigid element r in a semirigid GCD domain is a valuation element. (For the rigid element r show as in the proof of (2) $\Rightarrow$ (3) of Corollary 1 that $P(r) = \{x \in D|(x,r)_v \neq D\}$ is a maximal t-ideal of D and establish that $rD_{P(r)} \cap D = rD$, using the fact that r is homogeneous and that $D_{P(r)}$ is a valuation domain.) That a semirigid GCD domain is a URD is now immediate. (8) $\Rightarrow$ (7). This is direct since a VFD is semi-t-packed, according to [**?**].

(1) $\Rightarrow$ (9). This is direct since a semirigid GCD domain is a semirigid pre-Schreier domain, and a URD in which r is rigid $\Leftrightarrow$ r is a packet.

(9) $\Rightarrow$ (1). This follows from the fact that D is a pre-Schreier domain in which every nonzero non unit is expressible as product of mutually coprime packets each of which is rigid. (One may look at the product of two non-coprime rigid elements r, s. Then $rs = p_1p_2.., .p_n$ where p_i are mutually coprime packets. Now $r|p_1p_2.., .p_n$ implies $r = r_1r_2.., .r_n$ where $r_i|p_i$. But as r is rigid and so cannot have mutually coprime non unit factors, we conclude that $r|p_i$ for exactly one i, say $r|p_1$. Similarly s divides exactly one of p_i and that cannot be other than p_1. So s is coprime with $p_2, ...p_n$. Now $s = (p_1/r)p_2...p_n$ and s, p_i coprime for $i \geq 2$ forces s to divide (p_1/r). But then it is easy to conclude that $rs = p_1$. Thus in D the property $*$ holds and Corollary 1 applies.)

(1) $\Rightarrow$ (10). This is direct as a semirigid GCD domain is a PSP domain in which products of non-coprime rigid elements are rigid.

(10) $\Rightarrow$ (1). This follows because D has PSP and so "products pairs of non-coprime rigid elements being rigid" implies "products of pairs of non-v-coprime rigid elements being rigid" which is property $*$ and Corollary 1 applies.

(1) $\Rightarrow$ (11). This follows because in a semirigid GCD domain every rigid element is a valuation element, as shown in the proof of (1) $\Rightarrow$ (8), and hence a packed element. Moreover for a pair of rigid elements r, s in a semirigid GCD domain $(r, s)_v = D$ or r and s are comparable.

(11) $\Rightarrow$ (1). A semi-t-packed domain with the given condition is clearly a semirigid pre-Schreier domain in which every nonzero non unit is a product of mutually coprime rigid elements and so the proof of (9) $\Rightarrow$ (1) applies.

(1) $\Rightarrow$ (12). This follows directly as a semirigid GCD domain is a GCD domain.

(12) $\Rightarrow$ (1). (Use proof of Theorem 2.3 or proceed as follows.) For this we first note that every rigid element r of D is primal. This is because letting $r|xy$ we get $r = r_1r_2$ where $r_1 = (r, x)_v$. So $r = r_1r_2$ and $x = r_1x_2$ where $(r_2, x_2)_v = D$. Now as $r_2|x_2y$, and $(r_2, x_2)_v = D$, we conclude that $r_2|y$. Using the facts that D is semirigid, that products of primals are primal and that units are primal we conclude that all nonzero elements of D are primal and that D is pre-Schreier. But then every rigid element is homogeneous (i.e. belongs to a unique maximal t-ideal). Once that done, note that $(r, x)_v = (r, x)_t$. Now with some effort one can show that, in the terminology of [9], every rigid element of D is t-f-homog and D a t-f-Semi Homogeneous (t-f-SH) domain. Setting $* = t$ in Theorem 17 of [9] we conclude that D is a GCD independent ring of Krull type and hence a semirigid GCD domain.

(1) $\Rightarrow$ (13). This is direct as a semirigid GCD domain is a VFD with rigid elements being the valuation elements with all those properties.

(13) $\Rightarrow$ (12). This is obvious because in a VFD valuation elements are rigid and every valuation element having GCD translates to $(r,x)_v = hD$ because a VFD is Schreier and in a Schreier domain coprime is v-coprime (Lemma 2.1 of [28]). $\square$

While the jury is out on whether a VFD is a semirigid GCD domain or not, it is patent that generally a semi tenacious domain is not a semirigid GCD domain, nor is a t-SPD a semirigid GCD domain. However each collapses into a semirigid GCD domain if D is of t-dimension one, i.e., if every maximal t-ideal of D is of height one. Yet before we show that a bit of introduction is in order.

Call a domain D a generalized UFD (GUFD) if every nonzero non unit of D is expressible as a finite product of prime quanta. It was shown in [23] and later in [3], that every nonzero non unit of a GUFD is uniquely expressible as a finite product of mutually coprime prime quanta. We showed a GUFD to be a GCD domain that was also a generalized Krull domain (GKD), that is a domain D such that (a) D is a locally finite intersection of localizations at all height one primes of D and (b) D_P is a valuation domain for each height one prime of D. It so transpired that later, in [6], domains with just the (a) part were studied as weakly Krull domains. A weakly factorial domain D is a weakly Krull domain with $Cl_t(D)$ trivial. And as noted on page 350 of [7], just above Corollary 3.8, a weakly Krull domain that is a GCD domain is a GUFD.

Proposition 10. *The following are equivalent for an integral domain D : (1) D is a weakly factorial GCD domain, (2) D is VFD of t-dimension 1, (3) D is a generalized Krull GCD domain, (4) D is a GUFD, (5) D is a weakly Krull GCD domain, (6) D is a semi tenacious domain of t-dimension 1, (7) D is semi-t-packed with t-dim$(D) = 1$ (8) D is a semirigid GCD domain with t-dimension one.*

Proof. (1) $\Leftrightarrow$ (2) follows from [15, Corollary 1.9]. Next (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Theorem 10 of [3]. (3) $\Rightarrow$ (5) because a generalized Krull domain is a weakly Krull domain and (5) $\Rightarrow$ (3) follows from the observation that D weakly Krull implies that $D = \cap_{P\in X^1(D)} D_P$ where the intersection is locally finite. Next if D is GCD and P of height one, then D_P is a rank one valuation. But then D is a generalized Krull domain. Now (4) $\Rightarrow$ (6) because a GUFD is weakly Krull and hence of t-dimension one via [6, Lemma 2.1]. Next, by definition, a GUFD is tenacious and thus tenacious of t-dimension one. Next comes (6) $\Rightarrow$ (4). By definition a semi tenacious domain is a pre-Schreier domain and being power rigid every tenacious element is pre-homogeneous. But by Proposition 6 every tenacious element is homogeneous. Let h be a tenacious element in the semi tenacious domain of t-dimension one and let $P(h) = \{x \in D | (h,x)_v \neq D|$ be the maximal t-ideal containing h. Now let Q be the minimal prime of h. Then Q is of height one and hence a maximal t-ideal, because D is of t-dimension one. Thus $\sqrt{h} = P(h)$. Finally let k be a non unit factor of h. Then $k \in P(h)$ and $\sqrt{k} = P(k) = P(h)$. Consequently $h|k^n$ for some natural number n. But this forces h to be a prime quantum. Thus every tenacious element is a prime quantum. Whence D is a GUFD. Next, as a t-packed element is tenacious we have (7) $\Rightarrow$ (6) and we have already shown that (6) $\Rightarrow$ (4). Next for (4) $\Rightarrow$ (7) note that a GUFD is of t-dimension one and semi-t-packed. Finally (4) $\Rightarrow$ (8) because a GUFD is semirigid GCD and of t-dimension one. For (8) $\Rightarrow$ (4), note that if r is a rigid element in a semirigid GCD domain of t-dimension one and if h, k are non unit factors of r^n. Then h and k are non-coprime. (For if they were coprime then $GCD(h, r)$ and $GCD(k, r)$ would be coprime. Forcing $GCD(h, r) = 1$ or $GCD(k, r) = 1$. Yet neither would be possible as $h, k|r^n$.) Thus every pair of non unit factor r^n is comparable and r is power rigid. But that makes r tenacious. Now complete the proof as in the proof of (6) $\Rightarrow$ (4). $\square$

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