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ON $\star$-POTENT DOMAINS AND $\star$-HOMOGENEOUS IDEALS

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Dedicated to Dan Anderson

ABSTRACT. Let $\star$ be a star operation of finite character. Call a $\star$-ideal I of finite type a $\star$-homogeneous ideal if I is contained in a unique maximal $\star$-ideal $M = M(I)$. A maximal $*$-ideal that contains a $\star$-homogeneous ideal is called potent and the same name bears a domain all of whose maximal $\star$-ideals are potent. One among the various aims of this article is to indicate what makes a $\star$-ideal of finite type a $\star$-homogeneous ideal, where and how we can find one, what they can do and to direct to sources that indicate how this notion came to be.

1. INTRODUCTION

Let $\star$ be a finite character star operation defined on an integral domain D throughout. (A working introduction to star operations, and the reason for using them, will follow.) Call a nonzero finitely generated ideal I a $\star$-homogeneous ideal, if I is contained in a unique maximal $\star$-ideal. The notion of a $\star$-homogeneous ideal has figured prominently in describing unique factorization of ideals and elements in [5] and it seems important to indicate some other properties and uses of this notion and notions related to it. Call a $\star$-maximal ideal M $\star$-potent if M contains a $\star$-homogeneous ideal and call a domain D $\star$-potent if each of the $\star$-maximal ideals of D is $\star$-potent. The aim of this article is to study some properties of $\star$-homogeneous ideals and of $\star$-potent domains. We show for instance that while in a $\star$-potent domain every proper $\star$-ideal of finite type is contained in some $I^\star$ for I a $\star$-homogeneous ideal, the converse may not be true. We shall also indicate how these concepts can be put to use. Before we elaborate on that, it seems pertinent to give an idea of our main tool, the star operations. Indeed, the rest of what we plan to prove will be included in the plan of the paper after the introduction to star operations.

1.1. **Introduction to star operations.** Let D be an integral domain with quotient field K , throughout. Let $F(D)$ be the set of nonzero fractional ideals of D, and let $f(D) = \{A \in F(D) | A \text{ is finitely generated}\}$. A star operation $\star$ on D is a closure operation on $F(D)$ that satisfies $D^\star = D$ and $(xA)^\star = xA^\star$ for $A \in F(D)$ and $x \in K = K\backslash\{0\}$. With $\star$ we can associate a new star-operation $\star_s$

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given by $A \mapsto A^{\star_s} = \cup\{B^\star | B \subseteq A, B \in f(D)\}$ for each $A \in F(D)$. We say that $\star$ has finite character if $\star = \star_s$. Three important star-operations are the d-operation $A \mapsto A_d = A$, the v-operation $A \mapsto A_v = (A^{-1})^{-1} = \cap\{Dx | Dx \supseteq A, x \in K\}$ where $A^{-1} = \{x \in K : xA \subseteq D\}$ and the t-operation $t = v_s$. Here d and t have finite character. A fractional ideal A is a $\star$-ideal if $A = A^\star$and a $\star$-ideal A is of finite type if $A = B^\star$ for some $B \in f(D)$. If $\star$ has finite character and $A^\star$ is of finite type, then $A^\star = B^\star$ for some $B \in f(D)$, $B \subseteq A$. A fractional ideal $A \in F(D)$ is $\star$-invertible if there exists a $B \in F(D)$ with $(AB)^\star = D$; in this case we can take $B = A^{-1}$. For any $\star$-invertible $A \in F(D), A^\star = A_v$. If $\star$ has finite character and A is $\star$-invertible, then $A^\star$ is a finite type $\star$-ideal and $A^\star = A_t$. Given two fractional ideals $A, B \in F(D)$, $(AB)^\star$ denotes their $\star$-product. Note that $(AB)^\star = (A^\star B)^\star = (A^\star B^\star)^\star$. Given two star operations $\star_1$ and $\star_2$ on D, we write $\star_1 \leq \star_2$ if $A^{\star_1} \subseteq A^{\star_2}$ for all $A \in F(D)$. So $\star_1 \leq \star_2 \Leftrightarrow (A^{\star_1})^{\star_2} = A^{\star_2} \Leftrightarrow (A^{\star_2})^{\star_1} = A^{\star_2}$ for all $A \in F(D)$.Indeed for any finite character star-operation $\star$ on D we have $d \leq \star \leq t$.

For a quick introduction to star-operations, the reader is referred to [20, Sections 32, 34] or [40], for a quick review. For a more detailed treatment see Jaffard [27]. These days star operations are being used to define analogues of various concepts. The trick is to take a concept, e.g., a PID and look for what the concept would be if we require that for every nonzero ideal I, $I^\star$ is principal and voila! You have several concepts parallel to that of a PID. Of these t-PID turns out to be a UFD. Similarly a v-PID is a completely integrally closed GCD domain with the property that for each nonzero ideal we have A_v principal. A t-Dedekind domain, on the other hand is a Krull domain and a v-Dedekind domain is a domain with the property that for each nonzero ideal A we have A_v invertible see e.g [18]. So when we prove a result about a general star operation $\star$ the result gets proved for all the different operations, d, t, v etc., in some form. Apart from the above, any terminology that is not mentioned above, is either standard or, will be introduced at the point of entry of a concept.

Suppose that $\star$ is a finite character star-operation on D. Then a proper $\star$-ideal is contained in a maximal $\star$-ideal and a maximal $\star$-ideal is prime. We denote the set of maximal $\star$-ideals of D by $\star$-$Max(D)$. We have $D = \cap D_P$ where P ranges over $\star$-$Max(D)$.From this point on we shall use $\star$ to denote a finite type star operation. Call D of finite $\star$-character if for each nonzero non unit x of D, x belongs to at most a finite number of maximal $\star$-ideals. It may be noted that while we talk in terms of general star operations, our main focus will be the d, t and v-operations. Apart from the introduction there are three sections in this paper. In section 2 we talk about $\star$-homogeneous ideals, and $\star$-potent domains. We characterize $\star$-potent domains in this section, show that if D is of finite $\star$-character then D must be $\star$-potent, and characterize domains of finite $\star$-character, giving a new proof. In section 3, we show how creating a suitable definition of a $\star$-homogeneous ideal will create a theory of unique factorization of ideals. Call an element $r \in D$ $\star$-f-rigid ($\star$-factorial rigid), if rD is a $\star$-homogeneous ideal such that every proper $\star$-homogeneous ideal containing r is principal. Also call a $\star$-potent maximal $\star$-ideal M (resp., domain D) $\star$-f-potent if M (resp., every maximal $\star$-ideal of D) contains a $\star$-f-rigid element. We show that over a $\star$-f-potent domain a primitive polynomial f is super primitive i.e. if A_f, the content of f, is such that the generators of f have no non unit common factor then $(A_f)_v = D$ and indicate how to construct atomless domains. In this section we also offer a seamless patch to remove an error in the proof of a result in a paper by Kang [28] and show that D is t-super potent if and only if $D[X]_S$ is t-f-potent, where X is an indeterminate and $S = \{f \in [X] | (A_f)_v = D\}$. Here a $\star$-homogeneous ideal I is called $\star$-super homogeneous if every $\star$-homogeneous ideal J with $J^\star \supseteq I$ is $\star$-invertible. We also show, by way of constructing more examples, in this section that if L is an extension of K the quotient field of D and X an indeterminate over D then D is t-f-potent if and only if $D + XL[X]$ is.

2. $\star$-homogeneous ideals

Work on this paper started in earnest with the somewhat simple observation that if D is $\star$-potent then every nonzero non unit $x \in D$ is contained in $I^\star$ for some $\star$-homogeneous ideal I. The proof goes as follows: Because x is a nonzero non unit, xD is a proper $\star$-ideal and so must be contained in some maximal $\star$-ideal M. Now as D is $\star$-potent $M = M(I)$ for some $\star$-homogeneous ideal I. Consider $J = (I, x)$ and note that $(I, x)^\star \neq D$ because $x \in M$ and (I, x) is contained in a unique maximal $\star$-ideal and this makes J a $\star$-homogeneous ideal.

This leads to the question: If D is a domain with a finite character star operation $\star$ defined on it in such a way that every nonzero non unit x of D is contained in $I^\star$ for some $\star$-homogeneous ideal I of D, must D be $\star$-potent?

This question came up in a different guise as: when is a certain type of domain $*_s$-potent for a general star operation $*$ in [36] and sort of settled in a tentative fashion in Proposition 5.12 of [36] saying, in the general terms being used here, that: Suppose that D is a domain with a finite character star operation $\star$ defined on it. Then D is $\star$-potent provided (1) every nonzero non unit x of D is contained in $I^\star$ for some $\star$-homogeneous ideal I of D and (2) for M, $M_\alpha \in \star\text{-}max(D)$, $M \subseteq \cup M_\alpha$ implies $M = M_\alpha$ for some α.

The proof is something like: By (1) for every nonzero non unit x there is a $\star$-homogeneous ideal I_x with $(I_x)^\star$ containing x and so $x \in M(I_x)$. So $M \subseteq \cup M(I_x)$ and by (2) M must be equal to $M(I_x)$ for some x.

Thus we have the following statement.

Theorem 2.1. *Let $\star$ be a finite character star operation defined on D. Then D is $\star$-potent if D satisfies the following: (1)every nonzero non unit x of D is contained in $I^\star$ for some $\star$-homogeneous ideal I of D and (2) For M, $M_\alpha \in \star\text{-}max(D)$, $M \subseteq \cup M_\alpha$ implies $M = M_\alpha$ for some α.*

Condition (2) in the statement of Theorem 2.1 has had to face a lot of doubt from me, in that, is it really necessary or perhaps can it be relaxed a little?

The following example shows that condition (2) or some form of it is here to stay.

Example 2.2. It is well known that the ring $\mathcal{E}$ of entire functions is a Bezout domain [20, Exercise 18, p 147]. It is also easy to check that a principal prime in a Bezout domain is maximal. Now we know that a zero of an entire function determines a principal prime in $\mathcal{E}$ and that the set of zeros of a nontrivial entire function is discrete, including multiplicities, the multiplicity of a zero of an entire function is a positive integer [22, Theorem 6]. Thus each nonzero non unit of $\mathcal{E}$ is expressible as a countable product of finite powers of distinct principal primes of height one. For the identity star operation d, certainly defined on $\mathcal{E}$, only an ideal I generated by a power of a principal prime can be d-homogeneous. For if I is d-homogeneous, then $I = (x_1, ..., x_n)^d = x\mathcal{E}$ a principal ideal and hence a countable product of distinct primes. So I can only belong to a unique principal prime and has to be a finite prime power, to be d-homogeneous. To see that $\mathcal{E}$ falls foul of Theorem 2.1, let's put $S = \{p | p$ a prime element in $\mathcal{E}\}$. Then for each non principal prime P of $\mathcal{E}$ we have $P \subseteq \cup_{p\in S} p\mathcal{E}$ because each element of P is divisible by some member(s) of S, but a non-principal prime cannot be contained in a principal prime.

Condition (2) in Theorem 2.1 may remind some readers of the following result of Smith [34]: A ring D satisfies $*$: (for P, P_α prime ideals of D, where $\alpha \in I$, $P \subset \cup P_\alpha$ implies $P \subseteq P_\alpha$ for some α) if and only if every prime ideal of D is the radical of a principal ideal in D. But of course the situations are different in that in Theorem 2.1 only maximal $\star$-ideals are considered whereas in Smith's theorem any union of primes containing a prime P has to deliver the prime containing P.

Once we know more about $\star$-homogeneous ideals we would know that rings do not behave in the same manner as groups do. To get an idea of how groups behave and what is the connection, the reader may look up [36]. Briefly, the notion of a $\star$-homogeneous ideal arose from the notion of a basic element of a lattice ordered group G (defined as $b > 0$ in G such that $(0, b]$ is a chain). A basis of G, if it exists, is a maximal set of mutually disjoint strictly positive basic elements of G. According to [12] an l.o. group G has a basis if and only if every strictly positive element of G exceeds a basic element. So if we were to take D being potent as having a basis (every proper $\star$-ideal of finite type being contained in $I^\star$ for a $\star$-homogeneous ideal I) then every proper $\star$-ideal of finite type being contained in $I^\star$ for a $\star$-homogeneous ideal does not imply that D is potent, as we have seen in Example 2.2.

We next tackle the question of where $\star$-homogeneous ideals can be found. Call D of finite $\star$-character if every nonzero non unit of D is contained in at most a finite number of maximal $\star$-ideals. Again, a domain of finite $\star$-character could be a domain of finite character (every nonzero non unit belongs to at most a finite number of maximal ideals) such as an h-local domain or a semilocal domain or a PID or a domain of finite t-character such as a Krull domain.

Proposition 1. *A domain D of finite $\star$-character is $\star$-potent.*

Proof. Let M be a maximal $\star$-ideal of D and let x be a nonzero element of M. If x belongs to no other maximal $\star$-ideal then xD is $\star$-homogeneous and M is potent. So let us assume that M, $M_1, M_2, ..., M_n$ is the set of all maximal $\star$-ideals containing x. Now consider the ideal $A = (x, y)$ where $y \in M \backslash (\cup M_i)$ for $i = 1, ..., n$. Obviously $A \subseteq M$ but $A \nsubseteq M_i$ because of y. Note that A cannot be contained in any maximal $\star$-ideal other than M, for if N were any maximal $\star$-ideal containing A then N would belong to $\{M, M_1, M_2, ..., M_n\}$ because of x. And N cannot be any of the M_i, because of y. Thus A is a $\star$-homogeneous ideal contained in M and M is $\star$-potent. Since M was arbitrary we have the conclusion. □

The above proof is essentially taken from the proof for part (2) of Theorem 1.1 of [2].

Now how do we get a domain of finite $\star$-character? The answer is somewhat longish and interesting. Bazzoni conjectured in [7] and [8] that a Prufer domain D is of finite character if every locally principal ideal of D is invertible. [23] were the first to verify the conjecture using partially ordered groups. Almost simultaneously [25] proved the conjecture for r-Prufer monoids, using Clifford semigroups of ideals and soon after I chimed in with a very short paper [41]. The ring-theoretic techniques used in this paper verified the Bazzoni conjecture for Prufer domains, verified it for a generalization of Prufer domains called Prufer v-multiplication domains (PVMDs) and helped prove Bazzoni-like statements for other, suitable, domains that were not necessarily Prufer/PVMD. (Recall that D is a PVMD if every t-ideal A of finite type of D is t-invertible i.e. $(AA^{-1})_t = D$.) In the course of verification of the conjecture I mentioned a result due to Griffin from [21] that says:

Theorem 2.3. *A PVMD D is of finite t-character if and only if each t-invertible t-ideal of D is contained in at most a finite number of mutually t-comaximal t-invertible t-ideals of D.*

As indicated in the introduction of [41] the set of t-invertible t-ideals of a PVMD is a lattice ordered group under t-multiplication and the order defined by reverse containment of the ideals involved and that the above result for PVMDs came from the use of Conrad's F-condition. Stated for lattice ordered groups Conrad's F-condition says: Every strictly positive element exceeds at most a finite number of mutually disjoint elements. This and Theorem 2.3, eventually led the authors of [16], to the following statement.

Theorem 2.4. *(cf. Theorem 1 of* [16]*) Let D be an integral domain, $\star$ a finite character star operation on D and let Γ be a set of proper, nonzero, $\star$-ideals of finite type of D such that every proper nonzero $\star$-finite $\star$-ideal of D is contained in some member of Γ . Let I be a nonzero finitely generated ideal of D with $I^\star \neq D$. Then I is contained in an infinite number of maximal $\star$-ideals if and only if there exists an infinite family of mutually $\star$-comaximal ideals in Γ containing I.*

This theorem catapulted the consideration of finiteness of character from Prufer-like domains to consideration of finiteness of $\star$-character in general domains. But, there was an error in the proof. There was no reason for the error as the the technique, Conrad's F-condition, involved in the proof of Theorem 2.4 had been used at other places such as [15], [33] and, later, [17] but there it was. The sad story has been included in [42] and I see no point in dwelling on it, especially because the statement of the theorem was correct, [10]. However, for the record I include below a proof of 2.4, that shows how such results should actually be proved.

Lemma 2.5. *A nonzero finitely generated ideal I is $\star$-homogeneous if and only if for each pair X, Y of proper $\star$-ideals of finite type containing I we have that $(X+Y)^\star$ is proper.*

Proof. See [16]. □

Remark 2.6. (1)Note that if A and B are proper $\star$-ideals such that $(A+B)^\star = D$ and if C is any proper $\star$-ideal containing B then $(A+C)^\star = D$, since $(A+C)^\star = (A+B+C)^\star$, (2) note also the change of definition of a $\star$-homogeneous ideal. In [16] an I was called $\star$-homogeneous if I is a $\star$-ideal of finite type that is contained in a unique maximal $\star$-ideal and in this paper, following [5] and [26], I call a finitely generated nonzero ideal I $\star$-homogeneous if I is contained in a unique $\star$-maximal ideal. As it is explained in [43], the two definitions are equivalent.

Theorem 2.7. *Let $\star$ be a finite type star operation defined on an integral domain D and let Π be a set of nonzero $\star$-ideals of finite type such that every nonzero $\star$-ideal is contained in at least one member of* Π. *Then D is of finite $\star$-character if and only if every $\star$-ideal of finite type of D is contained in at most a finite number of mutually $\star$-comaximal $\star$-ideals of finite type from* Π.

Proof. Let A be a $\star$-ideal of finite type of D. Lemma 2.5 ensures that if there are no two mutually $\star$-coprime $\star$-ideals of finite type(from Π) containing A, then A itself contains a $\star$-homogeneous ideal I such that $A = (I)^{\star}$, see [43]. Next we show (I) that every $\star$-ideal of finite type of D is contained in at least one $I^{\star} \in \Pi$ for a $\star$-homogeneous ideal I of D. For suppose that there is a $\star$-ideal A of finite type of D that is not contained in any $I^{\star}$ for a $\star$-homogeneous ideals I of D. Then obviously A is not $\star$-homogeneous. So there are at least two proper $\star$-ideals A_1, B_1 of finite type, such that $(A_1+B_1)^{\star} = D$ and $A \subseteq A_1, B_1$. Obviously, neither of A_1, B_1 is homogeneous. As B_1 is not $\star$-homogeneous there are at least two $\star$-comaximal proper $\star$-ideals B_{11}, B_{12} of finite type containing B_1. Now by Remark 2.6 A_1, B_{11}, B_{12} are mutually $\star$-comaximal proper $\star$-ideals containing A and by assumption none of these is $\star$-homogeneous. Let B_{123} and B_{22} be two $\star$-comaximal proper $\star$-ideals containing B_{12}. Then by Remark 2.6 and by assumption, A_1, B_{11}, B_{22}, B_{123} are proper mutually $\star$-comaximal $\star$-ideals containing A and none of these ideals is homogeneous, and so on. Thus at stage n we have a collection: $A_1, B_{11}, B_{22}, ...,$ $B_{nn}, B_{12...n,n+1}$ that are proper mutually $\star$-comaximal $\star$-ideals containing A and none of these ideals is homogeneous. Also as each of the ideals in the above list is contained in a member of Π, wich must be $\star$-comaximal with the other members of the list, we can conclude that list belongs to Π. Now the process is never ending and has the potential of delivering an infinite number of mutually $\star$-comaximal proper $\star$-ideals of finite type containing A, contrary to the finiteness condition. Whence the conclusion.

Let S be the set of all the $\star$-homogeneous ideals containing A and define a function $\varphi : S \to \star\text{-}Max(D)$ by $\varphi(H) = M(H)$ the unique maximal $\star$-ideal containing $H \in S$. Obviously φ is a well-defined function. Let $\{M_\alpha\} = \varphi(S)$ and note that $|\varphi(S)| < \infty$. For if we were to choose exactly one $\star$-homogeneous ideal H_α from each M_α then $\{H_\alpha\}$ is the set of mutually disjoint $\star$-homogeneous ideals containing A, which must be finite by the condition. Next claim that $\varphi(S) = \{M_1, M_2, ...M_n\}$ is the set of all the maximal $\star$-ideals containing A. For if $P \in \star\text{-}Max(D)\backslash\varphi(S)$ were a maximal $\star$-ideal containing A then there is $y \in P \backslash \cup_{i=1}^{n} M_i$. As $(A, y) \subseteq P$ we have $(A, y)^{\star} \neq D$, forcing (A, y) and hence A, in a homogeneous ideal $H \notin S$, a contradiction.

For the converse let A be a nonzero finitely generated ideal in a domain D of finite $\star$-character and let, by way of contradiction, $\{A_i|(A_i)^{\star} \neq D\}$. be infinitely many mutually $\star$-comaximal finitely generated nonzero ideals containing A. Since A_i are mutually $\star$-co-maximal the sets of maximal $\star$-ideals $\cup\{M_{\alpha i}\}$ contating $\{A_i\}$ would be infinite forcing A in an infinite set of distinct maximal $\star$-ideals. (Dan Anderson's help with the proof is gratefully acknowledged.) □

So, if we must construct a $\star$-homogeneous ideal we know where to go. Otherwise there are plenty of $\star$-potent domains, with one kind studied in [26] under the name $\star$-super potent domains.

3. What $\star$-homogeneous ideals can do

This much about $\star$-homogeneous ideals and potent domains leads to the questions: What else can $\star$-homogeneous ideals do? These ideals arose and figure prominently in the study of finite $\star$-character of integral domains, as we have seen above. The domains of $\star$-finite character where the $\star$-homogeneous ideals show their full force are the $\star$-Semi Homogeneous ($\star$-SH) Domains. Indeed as I indicate in section 4 of [42], these ideals have been with me ever since I wrote Chapter One of my doctoral thesis.

It turns out, and it is easy to see, that if I and J are two $\star$-homogeneous ideals that are similar, i.e. that belong to the same unique maximal $\star$-ideal (i.e. $M(I) = M(J)$ in the notation and terminology of [5]) then IJ is $\star$-homogeneous belonging to the same maximal $\star$-ideal. With the help of this and some auxiliary results it can then be shown that if an ideal K is a $\star$-product of finitely many $\star$-homogeneous ideals then K can be uniquely expressed as a $\star$-product of mutually $\star$-comaximal $\star$-homogeneous ideals. Based on this a domain D is called a $\star$-semi homogeneous ($\star$-SH) domain, if every proper nonzero principal ideal of D is expressible as a $\star$-product of finitely many $\star$-homogeneous ideals. It was shown in [5, Theorem 4] that D is a $\star$-SH domain if and only if D is a $\star$-h-local domain (D is a locally finite intersection of localizations at its maximal $\star$-ideals and no two maximal $\star$-ideals of D contain a common nonzero prime ideal.) Now if we redefine a $\star$-homogeneous ideal so that the $\star$-product of two similar, newly defined, $\star$-homogeneous ideals is a $\star$-homogeneous ideal meeting the requirements of the new definition, we have a new theory.

To explain the process of getting a new theory of factorization merely by producing a suitable definition of a $\star$-homogeneous ideal we give below one such theory.

Definition 3.1. Call a $\star$-homogeneous ideal I $\star$-axial homogeneous ($\star$-a-homogeneous), if for every finitely generated ideal J with $J^\star \supseteq I$ there is a positive integer n such that $(J^n)^\star$is contained in a proper principal ideal. Also call a domain D a $\star$-ASH ($\star$-axial-SH) domain if for every nonzero non unit x of D, xD is a $\star$-product of finitely many $\star$-axial homogeneous ideals and call D a $\star$-axial potent domain if every maximal $\star$-ideal of D contains a $\star$-axial homogeneous ideal.

Note that if I is a $\star$-axial homogeneous ideal and A is a $\star$-homogeneous ideal containing I, the proper principal ideal (d), that contains $(A^n)^\star$, must belong to $M(I)$. For if d belongs to any other maximal ideal N, then A will be contained in N too. Let I and J be two similar $\star$-axial homogeneous ideals. Since $M(I) = M(J)$ we conclude that $(IJ)^\star$ is a $\star$-homogeneous ideal. Next let F be a $\star$-homogeneous ideal such that F^* contains IJ. Then since $F + I$ contains I we have $((F + I)^n)^\star$ contained in a proper principal ideal (d), forcing $(F^n)^\star \subseteq (d)$.

Now that we have shown that the $\star$-product of two similar $\star$-axial homogeneous ideals is a $\star$-axial homogeneous ideal similar to them, we can develope the theory of $\star$-ASH domains exactly along the lines of $\star$-SH domains discussed in [5]. To establish that the theory of $\star$-ASH domains is not an empty theory we have the following result on offer. First let's recall that an integral domain D is a $\star$-AGCD domain if for every finite set of nonzero elements $x_1, ..., x_r \in D$ there is a positive integer $n = n(x_1, ..., x_r)$ such that $(x_1^n, ..., x_r^n)^\star$ is principal. Also, D is a pre-Schreier domain if $x|yz$ implies $x = rs$ where $r|y$ and $s|z$, for all $x, y, z \in D\backslash\{0\}$.

Proposition 2. *(a) In a $\star$-AGCD domain D every $\star$-homogeneous ideal is $\star$-axial and thus a $\star$-AGCD $\star$-SH domain is a $\star$-ASH domain, (b) Every t-homogeneous ideal in a Schreier domain D is a t-axial ideal and so a pre-Schreier t-SH domain is a t-ASH domain and (c) A weakly Krull domain D such that every height one prime is the radical of a principal ideal is a t/ASH domain.*

Proof. (a) Recall from Lemma 2.3 of [35] that for any integral domain D and for any finite set $x_1, ..., x_r \in D\backslash\{0\}, (x_1, ..., x_r)^{nr} \subseteq (x_1^n, ..., x_r^n) \subseteq (x_1, ..., x_r)^n$ and if $\star$ is defined on D, then $\star$ can be applied throughout. Now in a $\star$-AGCD D, $(x_1^n, ..., x_r^n)^\star = (d)$, for some $n = n(x_1, ..., x_r)$ and so for some $m(= nr)$ we have $((x_1, ..., x_r)^m)^\star$ contained in a proper principal ideal. Next if in a $\star$-AGCD domain D, I is a $\star$-homogeneous ideal then $I^\star \neq D$ and depending upon the choice of $x_1, ..., x_r \in D\backslash\{0\}$, such that $I = (x_1, ..., x_r)^\star$ and on $n = n(x_1, ..., x_r)$, we can find a positive integer m such that $(I^m)^\star$ is contained in a proper principal ideal, $(x_1^n, ..., x_r^n)^\star = (d)$, that is contained in $(I^n)^\star$ and hence in $M(I)$. The same can be said about any $\star$-homogeneous ideal J with $J^\star$ containing I, forcing I to be $\star$-axial. Consequently a $\star$-AGCD domain that is a $\star$-SH domain is a $\star$-ASH domain. (b) In Lemma 2.1 of [39] it was shown that if D is a Schreier domain and A a finitely generated ideal of D with $A_v \neq D$ then A must be contained in a proper principal ideal. As a Schreier domain is an integrally closed pre-Schreier domain and as the property of being integrally closed was not used in the proof of [39, Lemma 2.1] we conclude that the result holds for pre-Schreier domains. So in a pre-Schreier domain a t-homogeneous ideal I is such that I is contained in a proper principal ideal and so is the case of any t-homogeneous ideal containing I, making I a t-axial homogeneous ideal. Consequently a pre-Schreier domain that is a t-SH domain is a t-ASH domain. (c) Let us note that if D is a one dimensional quasi-local domain and if a, $x_1, ..., x_r$ are nonzero non units in D then $a|x_i^n$ for some positive integer n. Next let P be a height one prime ideal of the weakly Krull domain described in the statement of part (c) and let a be an element of P such that $P = \sqrt{(a)}$. Also let $(x_1, ..., x_r)_t$ be a t-homogeneous ideal contained in P. (Such ideals can be arranged as in the proof of Proposition 1.) Now $a|x_i^n$ for some n and all $i = 1, ..., r$ in D_P because D_P is one dimensional and $a|x_i^n$ in D_Q for every height one prime $Q \neq P$, because $a \notin Q$. But then $a|x_i^n$ in D and, by [35, Lemma 2.3], we have $((x_1, ..., x_r)^m) \subseteq (x_1^n, ..., x_r^n) \subseteq (a)$, which gives $((x_1, ..., x_r)^m)_t \subseteq (x_1^n, ..., x_r^n)_t \subseteq (a)$. But then every t-homogeneous ideal contained in a particular height one prime is a t-axial ideal. Now it is well known that for each nonzero non unit x in a weakly Krull domain D, $xD = (xD_{P_1} \cap D) \cap ... \cap (xD_{P_n} \cap D) = ((xD_{P_1} \cap D)...(xD_{P_n} \cap D))_t$ where each of $xD_{P_i} \cap D$ is a t-ideal of finite type contained only in P_i and hence is a t-axial homogeneous ideal. □

Next, each of the definitions of $\star$-homogeneous ideals can actually give rise to $\star$-potent domains in the same manner as the $\star$-super potent domains of [26]. In [26], for a star operation $\star$ of finite character, a $\star$-homogeneous ideal is called $\star$-rigid. The $\star$-maximal ideal containing a $\star$-homogeneous ideal I may be called a $\star$-potent maximal $\star$-ideal, as we have already done. Next we may call the $\star$-homogeneous ideal I $\star$-super-homogeneous if each $\star$-homogeneous ideal J with $J^\star$ containing I is $\star$-invertible and we may call a $\star$-potent domain D $\star$-super potent if every maximal $\star$-ideal I of D contains a $\star$-super homogeneous ideal. But then one can study $\star$-$\mathcal{A}$-potent domains where $\mathcal{A}$ refers to a $\star$-homogeneous ideal that corresponds to a particular definition. For example a $\star$-homogeneous ideal J is said to be of type 1 in [5] if $\sqrt{J} = M(J)$. So we can talk about $\star$-type 1 potent domains as domains each of whose maximal $\star$-ideals contains a $\star$-homogeneous ideal of type 1. The point is, to each suitable definition say $\mathcal{A}$ of a $\star$-homogeneous ideal we can study the $\star$-$\mathcal{A}$-potent domains as we studied the $\star$-super potent domains in [26]. Of course the theory corresponding to definition $\mathcal{A}$ would be different from that of other $\star$-potent domains. For example each of the maximal $\star$-ideal of the $\star$-type 1 potent domain would be the radical of a $\star$-homogeneous ideal etc. Now as it is usual we present some of the concepts that have some direct and obvious applications, stemming from the use of $\star$-homogeneous ideals. For this we select the $\star$-f-potent domains for a study.

3.1. **$\star$-f-potent domains.** Let $\star$ be a finite type star operation defined on an integral domain D which is not a field unless specifically stated, throughout this section. Call a nonzero non unit element r of D $\star$-factorial rigid ($\star$-f-rigid) if r belongs to a unique maximal $\star$-ideal and every finite type $\star$-homogeneous ideal containing r is principal. Indeed if r is a $\star$-f-rigid element then rD is a $\star$-f- homogeneous ideal and hence a $\star$-super homogeneous ideal, we may also call rD a $\star$-f-rigid ideal. So the terminology and the theory developed in [5] applies. Note here that every non unit factor s of a $\star$-f-rigid element r is again $\star$-f-rigid because of the definition. Note also that if r, s are similar $\star$-f-rigid elements (i.e. rD, sD are similar $\star$-f-homogeneous ideals) then rs is a $\star$-f-rigid element similar to r and s and so $r|s$ or $s|r$. Also if r is $\star$-f-rigid then r^n is $\star$-f-rigid for any positive integer n. Following definition 10 of [5] we say that D is a t-f-SH domain if every nonzero non-unit of D is expressible as a product of finitely many t-f-rigid (i.e. t-f-homogeneous) elements. By Theorem 17 of [5] a t-f-SH domain is a GCD domain.

Example 3.2. . Every power of a principal prime is a t-f-rigid element.

Call a maximal $\star$-ideal M $\star$-f-potent if M contains a $\star$-f-rigid element and call a domain D $\star$-f-potent if every maximal $\star$-ideal of D is $\star$-f-potent.

Example 3.3. . UFDs PIDs, Semirigid GCD domains, prime potent domains are all t-f-potent.

(domains in which every maximal t-ideal contains a prime element may be called prime potent. Indeed a prime element generates a maximal t-ideal [24, 13.5]. (So a domain in which every maximal t-ideal contains a prime element is simply a domain in which every maximal t-ideal is principal.)

The definition suggests right away that if r is $\star$-f-rigid and x any element of D then $(r, x)^\star = sD$ for some $s \in D$ and applying the v-operation to both sides we conclude that $GCD(r, x) = (r, x)_v$ of r exists with every nonzero element x of D and that for each pair of nonzero factors u, v of r we have $u|v$ or $v|u$; that is r is a rigid element of D, in Cohn's terminology [11]. Indeed it is easy to see, if necessary with help from [5], that a finite product of $\star$-f-rigid elements is uniquely expressible as a product of mutually $\star$-comaximal $\star$-f-rigid elements, up to order and associates and that if every nonzero non unit of D is expressible as a product of $\star$-f-rigid elements then D is, at least, a semirigid GCD domain of [37]. Also, as we shall show below, a t-f-potent domain of t-dimension one (i.e. every maximal t-ideal is of height one) is a GCD domain of finite t-character, but generally a t-f-potent domain is far from being a GCD domain. Before we delve into examples, let's prove a necessary result, by adjusting Theorem 4.12 of [13] and its proof to our situation.

Proposition 3. *Let D be an integral domain and let L be an extension field of the field of fractions K of D. Then each nonzero ideal F of $R = D + XL[X]$ is of the form $f(X)JR = f(X)(J + XL[X])$, where J is a D-submodule of L and $f(X) \in R$ such that $f(0)J \subseteq D$. If F is finitely generated J, is a finitely generated D-submodule of L.*

Proof. Indeed if $F = f(X)JR$ such that $f(0)J \subseteq D$, then F is an ideal of R. Next let $F = (\{f_\alpha\}_{\alpha\in\Gamma})$ and let F be such that $F \cap D \neq 0$. According to [14, Lemma 1.1], the following are equivalent for an ideal F of R: (1) F is such that $F \cap D \neq 0$, (2) $F \supseteq XL[X]$ and (3) $FL[X] = L[X]$. Further if any of these hold, then $F = F\cap D + XL[X] = (F\cap D)R$ and taking $f(X) = 1$, $J = F\cap D$ we have the stated form. Let's now consider the case when $IL[X] \neq L[X]$. In this case $IL[X] = f(X)L[X]$ where $f(X)$ is a variable polynomial of $L[X]$. Two cases arise: (i) $f(0) \neq 0$ and (ii) $f(0) = 0$. In case of (i) $f(X) = l + Xg(X) = l(1 + (X/l)g(X))$ where $l \in L\backslash\{0\}$. Let $f' = f(X)/l$ so that $f'(0) = 1$ and proceed as follows. Since $f'(X) \in R$, $F/f'(X)$ is a fractional ideal of R and hence an R-submodule. Because in $L[X]$, every element of F is divisible by $f'(X)$ $F = f'(X)(\{f_\alpha/f'\}_{\alpha\in\Gamma}) = f'(X)(\{r_{\alpha 0} + Xs_\alpha(X)\}_{\alpha\in\Gamma})$. Also because $f'(0) = 1$ we have $r_{\alpha 0} \in D$. So $F/f'(X) = (\{r_{\alpha 0} + Xs_\alpha(X)\}_{\alpha\in\Gamma})$ is an ideal of R. But since $F/f'(X)L[X] = L[X]$ we have $XL[X] \subseteq F/f'(X) = (\{r_{\alpha 0} + Xs_\alpha(X)\}_{\alpha\in\Gamma})$. Forcing $F/f'(X) = (\{r_{\alpha 0}\}_{\alpha\in\Gamma}) + XL[X]$, where $J = (\{r_{\alpha 0}\}_{\alpha\in\Gamma})$ is an ideal of D that can be termed as a D-submodule of L, [14, Lemma 1.1]. But then $F = f'JR$, as desired. Finally in case (ii) we have $FL[X] = f(X)L[X]$ where $f(0) = 0$ and we can take $f(X) = X^r g(X)$, where we can assume that $g(0) = 1$. Next every element and hence every generator of F is divisible by $f(X)$, in $L[X]$ and so we can write $F = f(X)(\{f_\alpha/f\}_{\alpha\in\Gamma}) = f(X)(\{l_\alpha h_\alpha(X)X^{s_\alpha}\})$, where $l_\alpha \in L\backslash\{0\}, h_\alpha(X) \in L[X]$ with $h_\alpha(0) = 1$ and $s_\alpha \geq 0$. Now $F/f(X) = (\{l_\alpha h_\alpha(X)X^{s_\alpha}\})$ has the now familiar property that $F/f(X)L[X] = L[X]$. But this time it means that (a) $(\{l_\alpha h_\alpha(X)X^{s_\alpha}\})\cap L \neq (0)$ and (b) there is a non-empty subset Δ of Γ such that $s_\delta = 0$ for all $\delta \in \Delta$. By (a) we have $XL[X] \subseteq F/f(X)$ and by (a) and (b) we have the D-submodule $(\{l_\delta\}_{\delta\in\Delta}) \subseteq F/f(X)$ of L. Thus $F/f(X) \supseteq (\{l_\delta\}_{\delta\in\Delta}) + XL[X]$. For the reverse containment let $h(X) \in F/f(X)$. If $h(0) = 0$, then $h(X) \in XL[X]$. If, on the other hand, $h(0) = g \neq 0$, then $h(X) = g + Xk(X)$ and as $f(X)h(X) \in F$ we have $f(X)(g + Xk(X)) \in F = (\{f_\alpha\}_{\alpha\in\Gamma})$. This gives $f(X)(g + Xk(X)) = r_{\alpha_1}(X)f_{\alpha_1}(X) + ... + r_{\alpha_n}(X)f_{\alpha_n}(X)$, where $r_{\alpha_i}(X) \in R$. Since, in $L[X]$, $f_{\alpha_i}(X)/f(X) = l_{\alpha_i}h_{\alpha_i}(X)X^{s_{\alpha_i}}$ we have $g + Xk(X) = \sum_{i=1}^n r_{\alpha_i}(X)\left(l_{\alpha_i}h_{\alpha_i}(X)X^{s_{\alpha_i}}\right) = \sum_{i=1}^n (r_{\alpha_{i0}} + Xs_{\alpha_i}(X))\left(l_{\alpha_i}h_{\alpha_i}(X)X^{s_{\alpha_i}}\right)$. Next since the left hand side of $g + Xk(X) = \sum_{i=1}^n (r_{\alpha_{i0}} + Xs_{\alpha_i}(X))\left(l_{\alpha_i}h_{\alpha_i}(X)X^{s_{\alpha_i}}\right)$ has a nonzero constant term, some of the α_i are in Δ. Without loss of generality we can assume that the first r of the s_{α_i} are zero and those α_i are δ_i. But then comparing the constants (by setting $X = 0$) we get $g = \sum_{i=1}^r (r_{\delta_{i0}})(l_{\delta_i}) \in (\{l_\delta\}_{\delta\in\Delta})$. Whence $g + Xk(X) \in (\{l_\delta\}_{\delta\in\Delta}) + XL[X]$ and $F/f(X) \subseteq (\{l_\delta\}_{\delta\in\Delta}) + XL[X]$. Thus $F/f(X) = (\{l_\delta\}_{\delta\in\Delta}) + XL[X] = ((\{l_\delta\}_{\delta\in\Delta}))R$ or $F = f(X)JR$ where J is a D-submodule of L such that $f(0)J \subseteq D$.

For the finitely generated case let's note that if $F = (f_1, f_2, ..., f_n)(D + XL[X])$ is an ideal of $R = D + XL[X]$ with $I = F \cap D \neq (0)$ then $F = (f_{10}, f_{20}, ..., f_{n0} + XL[X])(D + XL[X])$; where f_{i0} are the constant terms of f_i. This is because, if $F \cap D \neq (0)$ then $XL[X] \subseteq F$. Thus the constant terms are all contained in F. This gives $F \supseteq (f_{10}, f_{20}, ..., f_{r0} + XL[X])$. But for $f(X) \in F$ we have $f(X) = \sum a_i(X)f_i(X) = \sum a_{i0}f_{i0} + Xh(X) \subseteq f_{10}, f_{20}, ..., f_{n0} + XL[X]$. Thus we have $F = (f_{10}, f_{20}, ..., f_{n0})(D + XL[X]) = f(X)J(D + XL[X])$ where $f(X) = 1$ and $(f_{10}, f_{20}, ..., f_{n0}) = J$, an ideal of D. If on the other hand $I = F\cap D = (0)$, $FL[X] = f(X)L[X]$ where $f(X) \in L[X]$ and $f_i(X) = f(X)h_i(X)$ with $h_i(X) \in L[X]$. Now suppose that $f(0) \neq 0$. Then we can assume that $f(0) = 1$ and so $h_i(X) = (h_{i0} + Xk_i(X))$ where $h_{i0} \in D$ and so $F = f(X)(h_{10} + Xk_1(X), ..., h_{n0} + Xk_n(X))R$. Since

$(F/f(X))L[X] = L[X]$, $(F/f(X)) \cap D \neq (0)$. Resulting in $XL[X] \subseteq F/f(X) = (h_{10} + Xk_1(X), ..., h_{n0} + Xk_n(X))R$ and thus making $F/f(X) = (h_{10}, ..., h_{n0}) + XL[X])R$ or $F = f(X)(h_{10}, ..., h_{n0})R$, where $(h_{10}, ..., h_{n0}) = J$.

Finally if $f(0) = 0$, then $f(X)L[X] \subseteq XL[X]$ and so $F = f(X)J(D + XL[X])$ where J is a D-submodule of L as determined in the following. Since $f(0) = 0$ we can write $f(X) = X^r g(X)$, where $g(0) = 1$. This gives in return $F = (f_1, f_2, ..., f_n)(D + XL[X]$, $f_i = g(X)l_i X^{r+s_i} h_i(X)$ where $s_i \geq 0$. Thus we have $F/f(X) = \{l_1h_1(X)X^{s_1}, ..., l_nh_n(X)X^{s_n})$. But as $(F/f(X))L[X] = L[X]$, $F/f(X) = (l_1h_1(X)X^{s_1}, ..., l_nh_n(X)X^{s_n})$ must contain some $l \in L\backslash\{0\}$ and consequently some $s_i = 0$. This results in $XL[X] \subseteq (l_1h_1(X)X^{s_1}, ..., l_nh_n(X)X^{s_n})$ making $F/f(X) = ((l_1, l_2, ..., l_r) + XL[X])$, where $J = (l_1, l_2, ..., l_r)$ is a D-submodule of L because $F/f(X)$ is an R-module. But then $F = f(X)JR$, as claimed. □

Before we proceed further, it seems pertinent to give an idea of the prime and maximal t-ideals of $R = D + XL[X]$. As Proposition 3 indicates a general nonzero ideal I of R is of the form $I = f(X)FR$ where $f \in R$ and F is a D-submodule of L such that $f(0)F \subseteq D$. Moreover I can be of two types: (i) I such that $I \cap D \neq (0)$, which is equivalent to $IL[X] = L[X]$, by [14, Lemma 1.1] and (ii) I such that $I \cap D = (0)$. (And this, by [14, Lemma 1.1], is equivalent to $IL[X] \neq L[X]$ which indeed is the same as $IL[X] = f(X)L[X]$.) So there are two kinds of prime ideals. The prime and maximal ideals of type (i) are given as ideals P of R such that $P = (P \cap R) + XL[X]$ where P is maximal if and only if $P \cap R$ is, by [14, Theorem 1.3]. A prime ideal of type (ii) is either $XL[X]$, or a contraction from a prime of $L[X]$ of the form $f(X)L[X]$, via [14, Theorem 1.3, Lemmas 1.2, 1.5]. If indeed f is such that $f(0) \neq 0$ then by [14, Theorem 1.3, Lemma 1.5] $f(X)L[X]$ contracts to $f(X)R$ where $f(X)$ is irreducible and $f(0) = 1$ and $f(X)R$ is a maximal height one prime. So maximal ideals of type (ii) are precisely of height one and are principal. This means that if D is not a field, maximal ideals of type (ii) are principal of height one and so (maximal) t-ideals. This leaves maximal t-ideals of type (i). For that note that $D + XL[X]$ has the $D + M$ form. Thus if I is a nonzero ideal of D then $(I + XL[X])_v = I_v + XL[X] = I_v(D + XL[X])$, by [6, Proposition 2.4] and using this we can also conclude that $(I + XL[X])_t = I_t + XL[X] = I_t(D + XL[X])$. Thus if P is a maximal t-ideal of R with $P \cap D \neq (0)$, then $P = (P \cap D) + XL[X]$ where $(P \cap D)$ is a maximal t-ideal of D. On the other hand if $\wp$ is a maximal t-ideal of D then so is $P = \wp + XL[X]$. Indeed if D is a field, nonzero prime ideals of $R = D + XL[X]$ are either principal of the form $(1 + Xg(X))R$ or the prime $XL[X]$ and all are maximal of height one.

Lemma 3.4. *Let D be an integral domain (that is not a field) and let L be an extension field of the field of fractions K of D. Then (a) $d \in D\backslash(0)$ is a t-f-rigid element of D if and only if d is a t-f-rigid element of $R = D + XL[X]$ and (b) An element α of $XL[X]$ is t-f-rigid if and only if $\alpha = lX^r$, D is a valuation domain and $K = L$. (In part (b) "t-f-rigid" can be replaced by "d-f-rigid" or just "rigid".)*

Proof. Let d be a t-f-rigid element of D then dD is a t-f-rigid ideal, so any t-ideal of finite type, of D, containing dD is principal. Next consider $d \in D + XL[X]$. Any t-ideal of finite type F of R containing d intersects D and so has the form $(F \cap D) + XL[X]$, according to [14, Lemma 1.1]. Consequently F contains $dD + XL[X]$. We show that F is principal. For this let $F = (a_1 + Xf_1(X), ..., a_n + Xf_n(X)_t = ((a_1, ..., a_n) + XL[X])_v = ((a_1, ..., a_n)_v + XL[X])$. But $((a_1, ..., a_n)_v + XL[X]) =$

$F \supseteq dD + XL[X]$ forces $(a_1, ..., a_n)_v \supseteq dD$. Also dD being t-f-rigid, $(a_1, ..., a_n)_v$ must be principal, whence $F = dD + XL[X] = d(D + XL[X]) = dR$ is principal. Now note that according to [14], every prime ideal M of R that intersects D is of the form $M \cap D + XL[X]$ and using [6, Proposition 2.4] we can show that every maximal t-ideal M of $D + XL[X]$ that intersects D is of the form $M \cap D + XL[X]$ where $M \cap D$ is a maximal t-ideal of D and that, conversely, if m is a maximal ideal of D then $m + XL[X]$ is a maximal ideal of R. Thus, finally, if m is the unique maximal t-ideal of D containing d then $m + XL[X]$ is a maximal t-ideal of R containing d. Note that if N were another maximal t-ideal containing d then $N \cap D$ would be another maximal t-ideal of D containing d, a contradiction. Thus d is a t-f-rigid ideal of R. For (b), let $\alpha \in XL[X]$. Then $\alpha = lX^r g(X)$ where $g(0) = 1$ and $l \in L\backslash\{0\}$. So lX and X are t-f-rigid for every $l \in L\backslash\{0\}$ because $X, lX | \alpha^2$, at least. Thus $lX|X$ or $X|lX$, being two similar rigid elements. This forces $l \in D$ or $l^{-1} \in D$ for each $l \in L\backslash\{0\}$ and this makes $L = qf(D)$ and D a valuation domain. Also if $g(X) \neq 1$ we do not have $g(X)|X$, nor do we have $X|g(X)$. Thus $\alpha = lX^r$ as claimed. Conversely if D is a valuation domain and $L = qf(D)$ are as given, then $R = D + XK[X]$ is a Bezout domain and X belongs to a unique maximal ideal $M + XL[X]$, where M is the maximal ideal of D. Since R is Bezout, every finitely generated ideal containing X is principal, same with lX^r. □

Proposition 4. *Let D be an integral domain that is not a field and let L be an extension field of the field of fractions K of D. Then D is t-potent if and only if $R = D + XL[X]$ is. If D is a field, then $R = D + XL[X]$ is t-potent.*

Proof. Note that, according to [14, Lemma 1.2], every prime ideal P of R that is not comparable to $XL[X]$ contains an element of the form $1 + Xg(X)$, so must contain a prime element of the form $1 + Xg(X)$ and so must be a principal prime. Note also that D is not a field, so no homogeneous ideal of R is contained in $XL[X]$. We next show that a finitely generated ideal $F \nsubseteq XL[X]$ of R is t-homogeneous if and only if F is of the form $A + XL[X]$, where A is a t-homogeneous ideal of D or F is generated by a prime power of the form $(1 + Xh(X))^n$. For this note that if $F = A + XL[X]$ and A is contained in a unique maximal t-ideal P of D then $A + XL[X]$ is contained in the maximal t-ideal $P + XL[X]$ and any maximal t-ideal of R that contains $A + XL[X]$ also contains $P + XL[X]$. Also if $A + XL[X]$ is contained in a unique maximal t-ideal then the maximal t-ideal that contains $A + XL[X]$ whould have to be of the form $P + XL[X]$ where P contains A. Thus A is a t-homogeneous ideal of D if and only if $A + XL[X]$ is of R. Next, an ideal generated by a prime power is t-homogeneous anyway. Conversely let F be a finitely generated nonzero ideal of R. Then by Proposition 3, $F = f(X)(J + XL[X])$ where $f(X) \in R$. As F is not contained in $XL[X]$ there are two possibilities (i) $f(X) = 1$ forcing J to be a finitely generated ideal of D or (ii) $f(0) = 1$. If in addition F has to be t-homogeneous, then in case (i) F is contained in a prime ideal of the form $P + XL[X]$ and in case (ii) in a prime ideal incomparable with $XL[X]$, forcing $J = D$. In the first case $F = J + XL[X]$ where J is a t-homogeneous ideal belonging to P and in the second case $F = f(X)R$, where $f(X)R$, being homogeneous, is contained in a principal prime that contains an element of the type $1 + Xg(X)$ [14, Lemmas 1.2, 1.5] and so is a prime power of the desired type. If D is a field then $R = D + XL[X]$ is one dimensional. So all maximal ideals are t-ideals, of which one is $XL[X]$ and all the others are principal of the form $(1 + Xg(X))R$, where $1+Xg(X)$ is irreducible in $L[X]$. Indeed every finitely generated ideal of R contained entirely in $XL[X]$ is t-homogeneous belonging to $XL[X]$, in this case. (Ideals such as XR, (X,lX) etc.) □

Corollary 1. *Let D be an integral domain that is not a field and let L be an extension field of the field of fractions K of D. Then D is t-f-potent if and only if $R=D+XL[X]$ is. In case D is a field $R=D+XL[X]$ is t-f-potent if and only if $D=L$.*

Proof. Suppose that D is t-f-potent. As in the proof of Proposition 4 every maximal t-ideal P of R that is not comparable to $XL[X]$ contains an element of the form $1+Xg(X)$, so must contain a prime element of the form $1+Xg(X)$ and so must be a principal prime. Next the maximal t-ideals comparable to $XL[X]$ are of the form $P+XL[X]$ where P is a maximal t-ideal of D. Since D is t-f-potent P contains a t-f-rigid element, which is also a t-f-rigid element of R, by Lemma 3.4. So $P+XL[X]$ contains a t-f-rigid element of R. In sum, every maximal t-ideal of R contains a t-f-rigid element of R and so R is t-f-potent. Conversely suppose that R is t-f-potent. Then, as for each maximal t-ideal M of D, $\mathcal{M}=M+XL[X]$ is a maximal t-ideal of R, so must contain a t-f-rigid element r of R. We claim that there is a t-f-rigid $r\in M$. For if not and $r=g+Xh(X)\in\mathcal{M}$, then two cases arise (i) $g\neq 0$ and (ii) $g=0$. Case (i) is possible only if $Xh(X)=0$ and this forces $r\in M$. Next, by (b) of Lemma 3.4, case (ii) is possible only if $r=lX^r$, D is a valuation domain and $L=K$. But then every nonzero non-unit of D, is t-f-rigid and in $\mathcal{M}=M+XK[X]$. Thus each maximal t-ideal of D contains a t-f-rigid element of D. In case D is a field we proceed as follows. If $D=L$, then $R=L[X]$ which is a PID and so t-f-potent (and d-f-potent). If on the other hand $R=D+XL[X]$ is t-f-potent, with D a field, then X,lX must be t-f-rigid, for each $l\in L\backslash\{0\}$. But then $X|lX$ or $lX|X$ and in either case $l\in D$. □

Recall, from [1], that a GCD domain of finite t-character that is also of t-dimension 1 is called a generalized UFD (GUFD).

Example 3.5. If D is a UFD (GUFD, Semirigid GCD domain) that is not a field and L an extension of the quotient field K of D, then the ring $D+XL[X]$ is a t-f-potent domain.

The t-f-potent domains and their examples are nice but we must show that they have some useful properties. We have here the most striking property. Let X be an indeterminate over D. A polynomial $f=\sum a_iX^i$ is called primitive if its content $A_f=(a_0,a_1,...,a_n)$ is a primitive ideal, i.e., $(a_0,a_1,...,a_n)\subseteq aD$ implies a is a unit and super primitive if $(A_f)_v=D$. It is known that while a super primitive polynomial is primitive, a primitive polynomial may not be super primitive, see e.g. Example 3.1 of [4]. A domain D is called a PSP domain if each primitive polynomial f over D is superprimitive, i.e. if f being primitive implies $(A_f)_v=D$.

Proposition 5. *A t-f-potent domain D has the PSP property.*

Proof. Let $f=\sum a_iX^i$ be primitive i.e. $(a_0,a_1,...,a_n)\subseteq aD$ implies a is a unit and consider the finitely generated ideal $(a_0,a_1,...,a_n)$ in a t-f- potent domain D. Then $(a_0,a_1,...,a_n)$ is contained in a maximal t-ideal M associated with a t-f-rigid element r (of course $M=M(rD)$) if and only if $(a_0,a_1,...,a_n,r)_t=sD\neq D$. Since every maximal t-ideal of a t-f-potent domain is associated with a t-f-rigid element, we conclude that in a t-f-potent domain D, $f = \sum a_i X^i$ primitive implies that A_f is contained in no maximal t-ideal of D; giving $(A_f)_v = D$ which means that each primitive polynomial f in a t-f-potent domain D is actually super primitive. □

Now PSP implies AP i.e. every atom is prime, see e.g. [4]. So, in a t-f-potent domain every atom is a prime. If it so happens that a t-f-potent domain has no prime elements then the t-f-potent domain in question is atomless. Recently atomless domains have been in demand. The atomless domains are also known as antimatter domains. Most of the examples of atomless domains that were constructed were the so-called pre-Schreier domains, i.e. domains in which every nonzero non unit a is primal (is such that $(a|xy$ implies $a = rs$ where $r|x$ and $s|y)$. One example (Example 2.11 [4]) was laboriously constructed in [4] and this example was atomless and not pre-Schreier. As we indicate below, it is easy to establish a method of telling whether a t-f-potent domain is pre-Schreier or not.

Cohn in [11] called an element c in an integral domain D primal if (in D) $c|a_1a_2$ implies $c = c_1c_2$ where $c_i|a_i$. Cohn [11] assumes that 0 is primal. We deviate slightly from this definition and call a nonzero element c of an integral domain D primal if $c|a_1a_2$, for all $a_1, a_2 \in D\backslash\{0\}$, implies $c = c_1c_2$ such that $c_i|a_i$. He called an integral domain D a Schreier domain if (a) every (nonzero) element of D is primal and (b) D is integrally closed. We have included nonzero in brackets because while he meant to include zero as a primal element, he mentioned that the group of divisibility of a Schreier domain is a Riesz group. Now the definition of the group of divisibility $G(D)(= \{\frac{a}{b}D : a, b \in D\backslash\{0\}\}$ ordered by reverse containment) of an integral domain D involves fractions of only nonzero elements of D, so it's permissible to restrict primal elements to be nonzero and to study domains whose nonzero elements are all primal. This is what McAdam and Rush did in [31]. In [38] integral domains whose nonzero elements are primal were called pre-Schreier. It turned out that pre-Schreier domains possess all the multiplicative properties of Schreier domains. So let's concentrate on the terminology introduced by Cohn as if it were actually introduced for pre-Schreier domains.

Cohn called an element c of a domain D completely primal if every factor of c is primal and proved, in Lemma 2.5 of [11] that the product of two completely primal elements is completely primal and stated in Theorem 2.6 a Nagata type result that can be rephrased as: Let D be integrally closed and let S be a multiplicative set generated by completely primal elements of D. If D_S is a Schreier domain then so is D. This result was analyzed in [4] and it was decided that the following version ([4, Theorem 4.4] of Cohn's Nagata type theorem works for pre-Schreier domains.

Theorem 3.6. *(Cohn's Theorem for pre-Schreier domains). Let D be an integral domain and S a multiplicative set of D. (i) If D is pre-Schreier, then so is D_S . (ii) (Nagata type theorem) If D_S is a pre-Schreier domain and S is generated by a set of completely primal elements of D, then D is a pre-Schreier domain.*

Now we have already established above that if r is a t-f-rigid element then $(r, x)_v$ is principal for each $x \in D\backslash\{0\}$. But then $(r, x)_v$ is principal for each $x \in D\backslash\{0\}$ if and only if $(r) \cap (x)$ is principal for each $x \in D\backslash\{0\}$. So, r is what was called in [3] an extractor. Indeed it was shown in [3] that an extractor is completely primal. Thus we have the following statement.

Corollary 2. *Let D be a t-f-potent domain. Then D is pre-Schreier if and only if D_S is pre-Schreier for some multiplicative set S that is the saturation of a set generated by some t-f rigid elements.*

(Proof. If D is pre-Schreier then D_S is pre-Schreier anyway. If on the other hand D_S is pre-Schreier and S is (the saturation of a set) multiplicatively generated by some t-f- rigid elements. Then by Theorem 3.6, D is pre-Schreier.)

We may note here that if D_S is not pre-Schreier for some multiplicative set S, then D is not pre-Schreier. So the decision making result of Cohn comes in demand only if D_S is pre-Schreier. Of course in the Corollary 2 situation, the saturation S of the multiplicative set generated by all the t-f-rigid elements of D, leads to "if D_S is not pre-Schreier then D is not pre-Schreier for sure and if D_S is pre-Schreier then D cannot escape being a pre-Schreier domain".

Example 3.7. Let $D = \cap_{i=1}^{i=n} V_i$ be a finite intersection of distinct non-discrete rank one valuation domains with quotient field $K = qf(D)$, X an indeterminate over D and let L be a proper field extension of K. Then (a) $D + XL[X]$ is a non-pre-Schreier, t-f-potent domain and (b) $D+XL[X]_{(X)}$ is an atomless non-pre-Schreier, t-f-potent domain.

Illustration: (a) It is well known that D is a Bezout domain with exactly n maximal ideals, M_i [29], with $V_i = D_{M_i}$. Thus $D= \cap D_{M_i}$ and each of M_i being a t-ideal must, each, contain a t-homogeneous ideal by Proposition 1. $R = D + XL[X]$ is t-f-potent by Corollary 1. Now $D + XL[X]$ is not pre-Schreier for the following reason. The set $D^* = D\backslash(0)$ is a multiplicative set of R and $R_{D^*} = (D+XL[X])_{D^*}$ $= K + XL[X]$ which is an atomic domain and not a UFD if $K \neq L$ and an atomic pre-Schreier domain must be a UFD. For part (b) note that since $R = D + XL[X]$ is t-f-potent, so is D and all the maximal (t-)ideals of $D + XL[X]_{(X)}$ are $M_i + XL[X]_{(X)}$ where M_i are the maximal ideala of D amd each M_i contains a t-f-rigid element. Finally as $(D + XL[X]_{(X)})_{D^*} = K + XL[X]_{(X)}$ which cannot be Schreier, being atomic and non-UFD.

One more result that can be added needs introduction to a neat construction called the Nagata ring construction these days. This is how the construction goes.

Let $\star$ be a star operation on a domain D, let X be an indeterminate over D and Let $S_\star = \{f \in D[X] | (A_f)^\star = D\}$. Then the ring $D[X]_{S_\star}$ is called the Nagata construction from D with reference to $\star$ and is denoted by $Na(D, \star)$. Indeed $Na(D, \star) = Na(D, \star_f)$

Proposition 6. ([28] *Proposition 2.1.) Let $\star$ be a star operation on R. Let $\star_f$ be the finite type star operation induced by $\star$. Let $S_\star = \{f \in D[X] | (A_f)_\star = D\}$. Then (1) $S_\star = D[X] \backslash \cup_{M\in\Gamma} M[X]$ where Γ is the set of all maximal $\star_f$-ideals of D. (Hence $S_\star$ is a saturated multiplicatively closed subset of $D[X]$.), (2) $\{M[X]_{S_\star}\}$ is the set of all maximal ideals of $[DX]_{S_\star}$.*

As pointed out in [19], proof of Part (1) of the following proposition has a minor flaw, in that for a general domain it uses a result ([20, 38.4]) that is stated for integrally closed domains. The fix offered in [19] is a new result and steeped in semistar operations. We offer, in the following, a simple change in the proof of [28, (1) Proposition 2.2.] to correct the flaw indicated above.

Proposition 7. *([28] Proposition 2.2.) Let T be a multiplicatively closed subset of $D[X]$ contained in $S_v = \{f \in D[X] | (A_f)_v = D\}$. Let I be a nonzero fractional ideal of D. Then (1) $(I[X]_T)^{-1} = I^{-1}[X]_T$, (2) $(I[X]_T)_v = I_v[X]_T$ and (3) $(I[X]_T)_t = I_t[X]_T$.*

(1) It is clear that $I^{-1}[X]_T \subseteq (I[X]_T)^{-1}$. Let $u \in (I[X]_T)^{-1}$. Since for any $a \in I \backslash \{0\}$ we have $(I[X]_T)^{-1} \subseteq a^{-1}D[X]_T \subseteq K[X]_T$ we may assume that $u = f/h$ with $f \in K[X]$ and $h \in T$. Then $f \in (I[X]_T)^{-1}$. Hence $fI[X]_T \subseteq D[X]_T$. Hence $bf \in D[X]_T$ for any $b \in I$. Now $bfg \in D[X]$ for some $g \in S_v$. So $(A_{bfg})_v \subseteq D$. By [32, Proposition 2.2.], $(A_{bfg})_v = (A_{bf}A_g)_v = (A_{bf})_v$, since $(A_g)_v = D$ and hence v-invertible. Therefore $bA_f \subseteq (bA_f)_v = (A_{bf})_v \subseteq D$ for any $b \in I$. Hence $A_f \subseteq I^{-1}$. Hence $f \in I^{-1}[X]$ and $f/h \in I^{-1}[X]_T$. Therefore $(I[X]_T)^{-1} = I^{-1}[X]_T$.

Theorem 3.8. *([28], Theorem 2.4.) Let $\star$ be a finite type star operation on D. Let I be a nonzero ideal of D. Then I is $\star$-invertible if and only if $I[X]_{S_\star}$ is invertible.*

Theorem 3.9. *([28], Proposition 2.14.) Let $\star$ be a star operation on D. Then any invertible ideal of $D[X]_{S_\star}$ is principal.*

Thus we have the following corollary.

Corollary 3. *Let I be a t-ideal of finite type of D. Then I is t-invertible if and only if $I[X]_{S_v}$ is principal.*

Proof. If $I[X]_{S_v}$ is principal, then $I[X]_{S_v}$ is invertible and so I is t-invertible, by Theorem 3.8. Conversely let F be a finitely generated ideal such that $F_t = I$. Then F is t-invertible and so, by Theorem 3.8, is $F[X]_{S_v}$ invertible and hence principal by Theorem 3.9. But then $F[X]_{S_v} = (F[X]_{S_v})_t = I[X]_{S_v}$. □

Lemma 3.10. *Let I be nonzero finitely generated ideal of D. Then $I_t[X]_{S_v}$ is d-homogeneous if and only if I is t-homogeneous. Consequently $I_t[X]_{S_v}$ is t-f-rigid if and only if I is t-super homogeneous.*

Proof. Let I be a t-homogeneous ideal of D. Then $I[X]_{S_v}$ is finitely generated and that $I_t[X]_{S_v}$ is a t-ideal of finite type is an immediate consequence of Proposition 7. If M is the unique maximal t-ideal containing I, then at least $M[X]_{S_v} \supseteq I[X]_{S_v}$. Suppose that $\mathcal{N}$ is another maximal ideal of $D[X]_{S_v}$ containing $I[X]_{S_v}$. But by Proposition 6, $\mathcal{N} = N[X]_{S_v}$ for some maximal t-ideal N of D. That is $N = D \cap N[X]_{S_v} \supseteq D \cap I[X]_{S_v} \supseteq I$. This forces $N = M$ and consequently $N[X]_{S_v} = M[X]_{S_v}$ making $I[X]_{S_v}$ d-homogeneous.

Conversely if $I[X]_{S_v}$ is d-homogeneous contained in a unique $M[X]_{S_v}$,suppose that N is another maximal t-ideal of D containing I. Then again $N[X]_{S_v} \supseteq ID[X]_{S_v}$ which is d-homogeneous, a contradiction unless $N = M$.

The consequently part follows from Corollary 3. □

Let's call a domain $\star$-f-r-potent if every maximal $\star$-ideal of D contains a $\star$-f-rigid element.

Proposition 8. *Let D be an integral domain with quotient field K, X an indeterminate over D and let $S_v = \{f \in D[X] | (A_f)_v = D\}$. Then (a) D is t-potent if and only if $D[X]_{S_v}$ is d-potent and (b) D is t-super potent if and only if $D[X]_{S_v}$ is d-f-r-potent*

Proof. (a) Suppose that D is t-potent. Let $M[X]_{S_v}$ be a maximal ideal of $D[X]_{S_v}$ and let I be a t-homogeneous ideal contained in M. By Lemma 3.10, $I_t[X]_{S_v}$ is d-homogeneous, making $M[X]_{S_v}$ d-potent. Conversely suppose that $D[X]_{S_v}$ is d-potent and let M be a maximal t-ideal of D. Then $M[X]_{S_v}$ is a maximal ideal of $D[X]_{S_v}$ and so contains a d-homogeneous ideal $\mathcal{I} = (f_1, f_2, ..., f_n)D[X]_{S_v}$. Now let $I = (f_1, f_2, ..., f_n)$. Then $\mathcal{I} = ID[X]_{S_v}$ and $I \subseteq (A_I)_t[X]_{S_v} \subseteq M[X]_{S_v}$, since $M[X]_{S_v}$ is a t-ideal and $f_i \in M[X]_{S_v} \cap D[X]$. This gives $\mathcal{I} = ID[X]_{S_v} \subseteq (A_I)_t[X]_{S_v} \subseteq M[X]_{S_v}$ making $(A_I)[X]_{S_v}$ another homogeneous ideal, contained in $M[X]_{S_v}$ and containing $\mathcal{I}$. But then $(A_I) \subseteq M$ is a t-homogeneous ideal, by Lemma 3.10. For part (b) use part (a) and Corollary 3. □

Next, another property that can be mentioned "off hand" is given in the following statement.

Theorem 3.11. *A t-f-potent domain of t-dimension one is a GCD domain of finite t-character.*

A domain of t-dimension one that is of finite t-character is called a weakly Krull domain. (D is weakly Krull if $D = \cap D_P$ where P ranges over a family $\mathcal{F}$ of height one prime ideals of D and each nonzero non unit of D belongs to at most a finite number of members of $\mathcal{F}$.) A weakly Krull domain D is dubbed in [5] as $\star$-weakly Krull domain or as a type 1 $\star$-SH domain. Here a $\star$-homogeneous ideal I is said to be of type 1 if $M(I) = \sqrt{I^\star}$ and D is a type 1 $\star$-SH domain if every nonzero non unit of D is a $\star$-product of finitely many $\star$-homogeneous ideals of type 1. In the following lemma we set $\star = t$.

Lemma 3.12. *A t-f-potent weakly Krull domain is a type* 1 *t-f-SH domain.*

Proof. A weakly Krull domain is a type 1 t-SH domain. But then for every pair I, J of similar homogeneous ideals $I^n \subseteq J_t$ and $J^m \subseteq I_t$ for some positive integers m, n. So J is a t-f-rigid ideal if I is and vice versa. Thus in a t-f-potent weakly Krull domain the t-image of every t-homogeneous ideal is principal. Whence every nonzero non unit of D is expressible as a product of t-f-rigid elements which makes D a t-f-SH domain and hence a GCD domain. □

Proof. of Theorem 3.11 Use Theorem 5.3 of [26] for $\star = t$ to decide that D is of finite t-character and of t-dimension one. Indeed, that makes D a weakly Krull domain that is t-f-potent. The proof would be complete once we apply Lemma 3.12 and note that a t-f-SH domain is a GCD domain and of course of finite t-character. □

Generally a domain that is t-f-potent and with t-dimension > 1, is not necessarily GCD nor of finite t-character.

Example 3.13. $D = Z + XL[[X]]$ where Z is the ring of integers and L is a proper extension of Q the ring of rational numbers. Indeed D is prime potent and two dimensional but neither of finite t-character nor a GCD domain.

There are some special cases, in which a t-f-potent domain is GCD of finite t-character.

i) If every nonzero prime ideal contains a t-f-rigid ideal. (Use (4) of Theorem 5 of [5]) along with the fact that D is a t-f-SH domain if and only if D is a t-SH domain with every t-homogeneous ideal t-f-rigid. Thus a t-f-potent domain of t-dim 1 is of finite character.

ii) If D is a t-f-potent PVMD of finite t-character that contains a set S multiplicatively generated by t-f-rigid elements of D and if D_S is a GCD domain then so is D. (This involves a straightforward use of Theorem 3.6 and the fact that a pre-Schreier PVMD is a GCD domain.)

I'd be doing a grave injustice if I don't mention the fact that before there was any modern day multiplicative ideal theory there were prime potent domains as Z the ring of integers and the rings of polynomials over them. It is also worth mentioning that there are three dimensional prime potent Prufer domains of finite character that are not Bezout. The examples that I have in mind are due to Loper [30]. These are non-Bezout Prufer domains whose maximal ideals are generated by principal primes. While those examples are so important that it's hard not to mention them, they are so intricate that one can't do justice to them in a few lines.

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