THE v-OPERATION IN EXTENSIONS OF INTEGRAL DOMAINS

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Abstract. An extension $D \subseteq R$ of integral domains is *strongly t-compatible* (resp., *t-compatible*) if $(IR)^{-1} = (I^{-1}R)_v$ (resp., $(IR)_v = (I_vR)_v$) for every nonzero finitely generated fractional ideal I of D. We show that strongly t-compatible implies t-compatible and give examples to show that the converse does not hold. We also indicate situations where strong t-compatibility and its variants show up naturally. In addition, we study integral domains D such that $D \subseteq R$ is strongly t-compatible (resp., t-compatible) for every overring R of D.

Key words and phrases: star operation, t-linked, t-compatible, strongly t-compatible, domain extensions, localizing system, QQR-domain, Prüfer domain.

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1. INTRODUCTION

Throughout this article, let D be an integral domain with quotient field K. Let $F(D)$ be the set of nonzero fractional ideals of D, $f(D)$ the set of nonzero finitely generated fractional ideals of D, and $I(D)$ the set of nonzero integral ideals of D. Recall that a *star operation* $*$ on D is a function $I \mapsto I^*$ on $F(D)$ with the following properties:

If $I, J \in F(D)$ and $0 \neq x \in K$, then

(i) $D^* = D$ and $(xI)^* = xI^*$;
(ii) $I \subseteq I^*$ and if $I \subseteq J$, then $I^* \subseteq J^*$; and
(iii) $(I^*)^* = I^*$.

For a quick review of properties of star operations, the reader may consult [23, Sections 32 and 34]. An $I \in F(D)$ is said to be a *$*$-ideal* if $I^* = I$, and a $*$-ideal I has *finite type* if $I = J^*$ for some $J \in f(D)$. A star operation $*$ is of *finite type* if $I^* = \bigcup\{J^* \mid J \in f(D) \text{ and } J \subseteq I\}$ for every $I \in F(D)$. To any star operation $*$, we

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can associate a star operation $*_s$ of finite type by defining $I^{*_s} = \bigcup\{J^* \mid J \in f(D)$ and $J \subseteq I\}$ for every $I \in F(D)$. Clearly $I^{*_s} \subseteq I^*$, and if I is finitely generated, then $I^* = I^{*_s}$.

Recall that for $I \in F(D)$, we have $I^{-1} = D :_K I = \{x \in K \mid xI \subseteq D\}$. The functions defined on $F(D)$ by $I \mapsto I_v = (I^{-1})^{-1}$ and $I \mapsto I_t = \bigcup\{J_v \mid J \in f(D)$ and $J \subseteq I\}$ are well known star operations, known as the v- and t-operations. An $I \in F(D)$ is *divisorial* or a *v-ideal* (resp., *t-ideal*) if $I_v = I$ (resp., $I_t = I$). By definition, the t-operation is the finite-type star operation associated to the v-operation.

Let D be a subring of an integral domain R. We call $D \subseteq R$ an *extension of integral domains* and call R an *overring* of D if $R \subseteq K$. We shall use the v- and t-operations extensively, and we shall assume a working knowledge of these operations. Following [15, 16], an integral domain R is said to be *t-linked* over its subring D if $I^{-1} = D$ implies that $(IR)^{-1} = R$ for every $I \in f(D)$. One reason for writing this article is the following comment in [42, page 443]. "We note that in each of the extensions $D \subseteq R$, discussed above, R is t-linked over D, i.e., for every $I \in f(D), I^{-1} = D$ implies $(IR)^{-1} = R$ ([15]). So in each case, there is a homomorphism $\theta : Cl_t(D) \longrightarrow Cl_t(R)$ defined by $\theta([I]) = [(IR)_t]$ ([3]). However, if R is t-linked over D, the extension $D \subseteq R$ may not satisfy any of (a)-(d) and may not satisfy any of the equivalent conditions. (These facts will be included in a detailed account in the promised article.)" The "equivalent conditions" mentioned in the quote are the equivalent conditions of [42, Proposition 2.6]. (The third author thanks Jesse Elliott for reminding him of that promise.) Our main task will be to provide the example(s) hinted at in the above quote. The rest of the plan will be presented after we have given sufficient introduction.

Using v_X- (resp., t_X-) to denote the v- (resp., t-) operation on an integral domain X, we shall prove and record the consequences of the following theorem.

Theorem 1.1. *Let R be an integral domain with quotient field L, and let D be a subring of R with quotient field K. Then the following statements are equivalent.*

1. $I_{v_D}R \subseteq (IR)_{v_R}$ *for every* $I \in f(D)$.
2. $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ *for every* $I \in f(D)$.
3. $I_{t_D}R \subseteq (IR)_{t_R}$ *for every* $I \in F(D)$.
4. $(IR)_{t_R} = (I_{t_D}R)_{t_R}$ *for every* $I \in F(D)$.
5. $(IR)_{v_R} = (I_{t_D}R)_{v_R}$ *for every* $I \in F(D)$.
6. *If I is an integral t-ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a t-ideal of D.*
7. *If I is a principal fractional ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a t-ideal of D.*

Moreover, if the following hypothesis holds:

(8) $R :_L IR = ((D :_K I)R)_{v_R}$ *for every* $I \in f(D)$,

then statements (1) - (7) all hold.

According to [8, Proposition 1.1], via [42, Proposition 2.6], conditions (1)-(6) are all equivalent and an extension $D \subseteq R$ of integral domains is called *t-compatible* if it satisfies any of (1)-(6) (e.g., $(IR)_{t_R} = (I_{t_D}R)_{t_R}$ for every $I \in F(D)$). (These are the equivalent conditions hinted at in the quote above.) More generally, as in [4], given star operations $*_D$ and $*_R$ on integral domains $D \subseteq R$, we say that $*_D$ and $*_R$ are *compatible* if $(IR)^{*_R} = (I^{*_D}R)^{*_R}$ for every $I \in F(D)$. We shall prove that (1)-(7) are all equivalent and that all of them are implied by the hypothesis (8). Our task will then be to give examples (i) that would show that none of (1)-(7) implies the hypothesis (8) of the theorem and examples (ii) that would give t-linked overrings that do not satisfy any of (1)-(7) and the conditions (a)-(d) of [42, page 443] which are listed below.

(a) $I^{-1}R = (IR)^{-1}$ for every $I \in f(D)$.
(b) $(I^{-1}R)_{v_R} = (IR)^{-1}$ for every $I \in f(D)$.
(c) $I^{-1}R = (IR)^{-1}$ for every $I \in F(D)$.
(d) $(I^{-1}R)_{v_R} = (IR)^{-1}$ for every $I \in F(D)$.

Clearly $(c) \Rightarrow (a) \Rightarrow (b)$ and $(c) \Rightarrow (d) \Rightarrow (b)$. In Theorem 3.7 (resp., Theorem 3.9), we determine the overrings of D that are characterized by condition (b) (resp., condition (d)). If D is integrally closed, then (a) holds for every overring R of D if and only if D is a Prüfer domain (Corollary 4.3).

Let us call an extension $D \subseteq R$ of integral domains *strongly t-compatible* if $D \subseteq R$ satisfies the hypothesis (8) of Theorem 1.1 (i.e., if $(IR)^{-1} = (I^{-1}R)_{v_R}$ for every $I \in f(D)$, or equivalently, condition (b) above holds) and call $D \subseteq R$ *v-compatible* if $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ for every $I \in F(D)$. Thus v-compatibility implies t-compatibility. In Section 2, we show that strong t-compatibility implies t-compatibility and give examples to show that the converse is not true. Section 3 is devoted to indicating the situations in which strong t-compatibility and some of its variants appear naturally, and we characterize the domain extensions where strong t-compatibility holds. Finally, in Section 4, we study integral domains D such that $D \subseteq R$ is t-compatible for every overring R of D and relevant notions.

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2. Proof of Theorem 1.1 and Examples

It would help if the readers knew some old notational conventions in case we use such notation or refer to articles that use that notation. We shall follow the convention that the inverses, and hence the v-operations, are with respect to the relevant rings (rings to whose (fractional) ideals the operation is applied). For example, if $I \in F(D)$, we shall use $(IR)^{-1}$ to mean $R :_L IR$, where $L = qf(R)$, $(I^{-1}R)_v$ to mean $(I^{-1}R)_{v_R}$, where $I^{-1} = D :_K I$, and $(I_vR)_v$ to mean $(I_{v_D}R)_{v_R}$ if no confusion is foreseen. In this section, we shall prove Theorem 1.1 and construct the examples. We start with a general result to cover some more ground.

Lemma 2.1. *Let $D \subseteq R$ be an extension of integral domains, and let $I \in F(D)$ such that $(IR)^{-1} = (I^{-1}R)_v$. Then $(IR)_v = (I_vR)_v$.*

Proof. Clearly $(IR)_v = ((IR)^{-1})^{-1} = ((I^{-1}R)_v)^{-1} = (I^{-1}R)^{-1}$ by hypothesis. Let $x \in I_v$; then $xI^{-1} \subseteq D$. Thus $xI^{-1}R \subseteq R$, and hence $x \in (I^{-1}R)^{-1} = (IR)_v$. Thus $I_v \subseteq (IR)_v$, which gives $I_vR \subseteq (IR)_v$, and hence $(I_vR)_v \subseteq (IR)_v$. Finally, since $I \subseteq I_v$, we have $IR \subseteq I_vR$, and thus $(IR)_v \subseteq (I_vR)_v$. Equality follows. □

Proof of Theorem 1.1. For the proof, we adopt the following approach. We note that the hypothesis (8) of Theorem 1.1 implies (2) by Lemma 2.1. Then we show that (1)-(7) are all equivalent. Using the fact that (1)-(6) are all equivalent (in light of [4, Section 4] and [8, Proposition 1.1]), we show (6) $\Rightarrow$ (7) $\Rightarrow$ (1) to complete the proof.

(6) $\Rightarrow$ (7) Suppose that $xR \cap D \neq (0)$ for some $x \in L$. Then $xR \cap D = (xR \cap R) \cap D$ is a t-ideal of D by (6) since $xR \cap R$ is an integral t-ideal of R.

(7) $\Rightarrow$ (1) Let $I \in f(D)$ (we may assume that $I \in I(D)$), and recall that $(IR)_v = \bigcap\{xR \mid x \in L \text{ and } IR \subseteq xR\}$. For every $x \in L$ such that $IR \subseteq xR$, we have $I \subseteq IR \subseteq (IR)_v \subseteq xR$, and thus $I \subseteq xR \cap D$. Since $xR \cap D$ is a t-ideal of D by (7) and I is finitely generated, we have $I_v = I_t \subseteq xR \cap D$. This gives $I_v \subseteq xR$ for every xR containing IR, and hence $I_v \subseteq (IR)_v$. Thus $I_vR \subseteq (IR)_v$. □

From Theorem 1.1, it follows that strong t-compatibility implies t-compatibility. For the remainder of the task at hand, let us ask: Do any of the conditions (1)-(7) imply the hypothesis (8) of Theorem 1.1? In other words, is it true that t-compatibility implies strong t-compatibility? To answer this question, in the negative, we use the following example.

Example 2.2. Let D be a one-dimensional local (Noetherian) domain that is not a DVR, and let R be its integral closure. Then R is t-linked over D ([15]). Let I be a nonzero nonprincipal ideal of D. Then $II^{-1}R = dR$ for some nonunit $d \in R$. Thus $I^{-1}R = dR(IR)^{-1} \neq (IR)^{-1}$, and as we are working in a PID, $I^{-1}R = (I^{-1}R)_v$. Hence $(IR)^{-1} \neq (I^{-1}R)_v$; so $D \subseteq R$ is not strongly t-compatible. That $D \subseteq R$ is t-compatible follows from the following result provided by Evan Houston [29] as a sleek alternative to our, somewhat cumbersome, earlier proof. For a specific example, let $D = \mathbb{Q}[[X^2, X^3]] \subseteq R = \mathbb{Q}[[X]]$ and $I = (X^2, X^3)$ be the maximal ideal of D. Then $(I^{-1}R)_v = R$ and $(IR)^{-1} = X^{-2}R$; so $(IR)^{-1} \neq (I^{-1}R)_v$.

Lemma 2.3. ([29]) Let D be a Noetherian integral domain with integral closure R, and let $I \in f(D)$. Then $(I_vR)_v = (IR)_v$.

Proof. Let $x \in (IR)^{-1}$; then $xI \subseteq R$. Since R is integral over D and I is finitely generated, there is a (necessarily) finitely generated ideal J of D with $xIJ \subseteq J$. Thus $xI_vJ_v \subseteq J_v$. Since J_v is finitely generated, this yields $xI_v \subseteq R$, i.e., $xI_vR \subseteq R$. Hence $x \in (I_vR)^{-1}$; so $(IR)^{-1} \subseteq (I_vR)^{-1}$. Taking inverses, we have $(I_vR)_v \subseteq (IR)_v$. The reverse inclusion is obvious; so $(I_vR)_v = (IR)_v$. □

Problem 2.4. Characterize the extensions $D \subseteq R$ of integral domains such that t-compatibility implies strong t-compatibility.

Example 2.2 is somewhat limited in that the extension $D \subseteq R$ that it provides is t-compatible. We next give an example that will do the job completely. For this, we need to quote an example from Mimouni [34, Example 2.10]. (The purpose of this example was to show that there are w-ideals that are not t-ideals.)

Example 2.5. Let $R = \mathbb{Q}(\sqrt{2})[[X, Y, Z]]$, where X, Y, Z are indeterminates over $\mathbb{Q}$. Then $R = \mathbb{Q}(\sqrt{2}) + M$ is a 3-dimensional integrally closed local (Noetherian) domain with maximal ideal $M = (X, Y, Z)R$. Now set $D = \mathbb{Q} + M$. Then $D = \mathbb{Q} + M$ is a local (Noetherian) domain with integral closure R ([10]). Since the maximal ideal M is common to both D and R, we have $M = MR$; and so for the prime ideals $P_1 = XR, P_2 = (X,Y)R$ of R, we have $P_1 \subsetneq P_2 \subsetneq M$. We claim that P_2 is not a t-ideal of D, while M is a t-ideal of D. This follows from the following observations. Since $ht_R(P_2) = 2$, we have $R = R\colon P_2 = (P_2\colon P_2) = D\colon P_2$. Similarly, $R = R\colon M = M\colon M = D\colon M$. Now, as $M^{-1} = D\colon M \supsetneq D$, we must have $M_v \subsetneq D$. But since $D = \mathbb{Q} + M$ is local, $M_v = M$. Next, since $R = D : P_2 = P_2^{-1} = M^{-1}$, we have $(P_2)_t = (P_2)_v = M_v = M$. But as $P_2 \subsetneq M$, we conclude that P_2 is not a t-ideal of D. From this example, we also have that every overring of D is t-linked because the only maximal ideal M of D is a v-ideal, and hence a t-ideal ([15, Theorem 2.6]).

Now reason as follows. By [8, Proposition 1.1] (or Theorem 1.1), an extension $D \subseteq R$ of integral domains is such that $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ for every $I \in f(D)$ if and only if every t-ideal of R contracts to a t-ideal of D or to (0). So if there is an R with $D \subseteq R$ an extension of integral domains and a nonzero t-ideal J of R with $J \cap D$ a nonzero non-t-ideal of D, then $(IR)_{v_R} \neq (I_{v_D}R)_{v_R}$ for some $I \in f(D)$. If R is also t-linked over D, then we have our example that was promised in [42].

Example 2.6. Let us go back to Example 2.5. In $D = \mathbb{Q} + M$, we have a chain of prime ideals $P_1 \subsetneq P_2 \subsetneq M$ for which P_2 is not a t-ideal. By [23, Corollary 19.7], there is a valuation overring T of D with maximal ideal M' and a chain of proper prime ideals $Q_1 \subsetneq Q_2 \subsetneq M'$ such that $Q_i \cap D = P_i$ and $M' \cap D = M$. It is well known that every nonzero ideal in a valuation domain is a t-ideal. Thus Q_2 is a t-ideal in T that contracts to a non-t-ideal P_2 in D; so $D \subseteq T$ is not t-compatible. As we noted in the explanation of the Mimouni example, T is t-linked over $D = \mathbb{Q}+M$. So as we reasoned above, there is a (finitely generated) nonzero ideal I of $D = \mathbb{Q}+M$, which we do not know anything about, such that $(IT)_{v_T} \neq (I_{v_D}T)_{v_T}$. Actually, $I = P_2$ fills the bill since $(P_2T)_{v_T} = (P_2T)_{t_T} = P_2T \subseteq Q_2$, while $((P_2)_{v_D}T)_{v_T} = (MT)_{v_T} = (MT)_{t_T} = MT \subseteq M'$, and $P_2T \neq MT$ since $P_2T \cap D = P_2$ and $MT \cap D = M$.

Finally, will Example 2.6 take care of (a)-(d)? Let us check. First off, the example we have is a Noetherian domain; so we only need to take care of (a) and (b). Next, every fractional ideal I of an integral domain D is expressible as $I = x^{-1}J$, where J is an integral ideal of D and $0 \neq x \in D$; so (a)-(d) can be stated for integral ideals because the denominators cancel out in each case.

From Lemma 2.1, we note that if $(IR)_v \neq (I_vR)_v$ for some $I \in I(D)$, then $(IR)^{-1} \neq I^{-1}R$ and $(IR)^{-1} \neq (I^{-1}R)_v$.

Example 2.7. Going back to Example 2.6, the ideal I of D for which $(IT)_{v_T} \neq (I_{v_D}T)_{v_T}$ is precisely the ideal for which $(IT)^{-1} \neq I^{-1}T$ and $(IT)^{-1} \neq (I^{-1}T)_v$. So Example 2.6 serves as an example of an extension $D \subseteq T$ of integral domains with T a t-linked overring of D for which (1)-(7) and (a)-(d) do not hold, and thus $D \subseteq T$ is not strongly t-compatible.

3. Applications and related results

Section 2 seems to indicate that the key assumption is that $D \subseteq R$ is an extension of integral domains such that $(IR)^{-1} = (I^{-1}R)_v$ for certain types of nonzero ideals I of D (that includes finitely generated ideals). In our next proposition, we replace the $I \in f(D)$ hypothesis in Theorem 1.1 by $I \in F(D)$.

Proposition 3.1. *Let R be an integral domain with quotient field L, and let D be a subring of R with quotient field K. Then statements (1) and (2) are equivalent, (2) $\Rightarrow$ (3), and statements (3)-(7) are equivalent.*

(1) $I_{v_D}R \subseteq (IR)_{v_R}$ *for every* $I \in F(D)$.
(2) $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ *for every* $I \in F(D)$.
(3) $I_{t_D}R \subseteq (IR)_{t_R}$ *for every* $I \in F(D)$.
(4) $(IR)_{t_R} = (I_{t_D}R)_{t_R}$ *for every* $I \in F(D)$.
(5) $(IR)_{v_R} = (I_{t_D}R)_{v_R}$ *for every* $I \in F(D)$.
(6) *If I is an integral t-ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a t-ideal of D.*
(7) *If I is a principal fractional ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a t-ideal of D.*

Moreover, if $R :_L IR = ((D :_K I)R)_{v_R}$ for every $I \in F(D)$, then statements (1) - (7) all hold.

Proof. Clearly (1) $\Leftrightarrow$ (2). That (2) $\Rightarrow$ (3) and statements (3)-(7) are all equivalent follow from Theorem 1.1.

For the "moreover" statement, suppose that $R{:}_L IR = ((D{:}_K I)R)_{v_R}$ for every $I \in F(D)$. Then (2) holds since $(IR)^{-1} = (I^{-1}R)_v \Rightarrow (IR)_v = (I_vR)_v$ by Lemma 2.1. Thus (1)-(7) all hold by the above remarks. □

Proposition 3.1 leaves one thinking "What if 't-ideal' is replaced by 'v-ideal' in (6) and (7) of Proposition 3.1?" The following result provides an answer.

Proposition 3.2. *Let $D \subseteq R$ be an extension of integral domains. Then the following statements are equivalent.*

(1) $I_{v_D}R \subseteq (IR)_{v_R}$ *for every* $I \in F(D)$.
(2) $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ *for every* $I \in F(D)$.
(3) *If I is an integral v-ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a v-ideal of D.*
(4) *If I is a principal fractional ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a v-ideal of D.*

Proof. Clearly (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (3) Let I be an integral v-ideal of R with $I \cap D \neq (0)$. Then $(0) \neq I \cap D \subseteq (I \cap D)_v \subseteq (I \cap D)_v R \subseteq ((I \cap D)R)_v \subseteq I_v = I$ by (1). Thus $(I \cap D)_v \subseteq I$ implies that $(I \cap D)_v \subseteq I \cap D$, which forces $I \cap D = (I \cap D)_v$ and the conclusion that $I \cap D$ is a v-ideal of D.

(3) $\Rightarrow$ (4) Let L be the quotient field of R, and let $0 \neq x \in L$ such that $xR \cap D \neq (0)$. Then $xR \cap D = (xR \cap R) \cap D$ is a v-ideal of D by (3) since $xR \cap R$ is an integral v-ideal of R.

(4) $\Rightarrow$ (1) Let $I \in I(D)$, and recall that $(IR)_v = \bigcap\{xR \mid x \in L \text{ and } IR \subseteq xR\}$. For every xR in the intersection, we have $I \subseteq xR \cap D$. Thus $I_v \subseteq xR \cap D$ since $xR \cap D$ is a v-ideal of D by (4). Hence $I_v \subseteq xR$, and thus $I_vR \subseteq xR$ for every xR such that $IR \subseteq xR$. Hence $I_vR \subseteq \bigcap\{xR \mid x \in L \text{ and } IR \subseteq xR\} = (IR)_v$ for every $I \in I(D)$, and thus for every $I \in F(D)$ as well. □

Recall that an extension $D \subseteq R$ of integral domains that satisfies the equivalent conditions of Proposition 3.2 (e.g., $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ for every $I \in F(D)$) is called v-compatible. Note that a v-compatible extension is t-compatible. The converse is true for Noetherian domains, but not in general (see Example 4.6). Thus (1)-(7) need not be equivalent in Proposition 3.1 (since (4) $\Rightarrow$ (2) need not hold), but (1)-(7) are equivalent in Theorem 1.1. Moreover, by Proposition 3.1, if $R :_L IR = ((D :_K I)R)_{v_R}$ for every $I \in F(D)$, then the extension $D \subseteq R$ is v-compatible. The converse is not true since in the Noetherian case, t-compatibility does not imply strong t-compatibility by Example 2.2.

Example 3.3. Let $R = Int(D)$, the ring of integer-valued polynomials over the integral domain D. Then the extension $D \subseteq R$ satisfies the "moreover" hypothesis of Proposition 3.1 precisely. That is, $(I(Int(D)))^{-1} = (I^{-1}Int(D))_v$ for every $I \in F(D)$. This result follows from [11, Lemma 3.1(1)(2)]. Indeed, Propositions 3.1 and 3.2 apply to this particular situation.

Apart from Example 3.3, most well known examples that could benefit from Proposition 3.1 fall under the category of extensions $D \subseteq R$ such that $(IR)^{-1} = I^{-1}R$, i.e., $I^{-1}R$ is divisorial, for every $I \in F(D)$. Because there are a number of known cases of this type, it seems in order to restate Proposition 3.1 for this special case.

Corollary 3.4. *Let R be an integral domain with quotient field L, and let D be a subring of R with quotient field K. If $R :_L IR = (D :_K I)R$ for every $I \in F(D)$, then the following statements hold.*

1. $(IR)_{v_R} = I_{v_D}R$ *for every* $I \in F(D)$.
2. $(IR)_{t_R} = I_{t_D}R$ *for every* $I \in F(D)$.
3. $(IR)_{v_R} = (I_{t_D}R)_{v_R}$ *for every* $I \in F(D)$.
4. *If I is an integral t-ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a t-ideal of D.*
5. *If J is an integral t-ideal (resp., v-ideal) of D, then JR is an integral t-ideal (resp., v-ideal) of R.*
6. *If I is a principal fractional ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a v-ideal of D.*

Proof. (1) $(IR)_v = ((IR)^{-1})^{-1} = (I^{-1}R)^{-1} = (I^{-1})^{-1}R = I_vR$ by hypothesis.

(1) $\Rightarrow$ (2) Recall that $(IR)_{t_R} = \cup\{J_{v_R} \mid J \in f(R) \text{ and } J \subseteq IR\}$. For each J in the definition, $J \subseteq J_1R$ for some finitely generated ideal $J_1 \subseteq I$, and so $J_{v_R} \subseteq (J_1R)_{v_R} = (J_1)_{v_D}R \subseteq I_{t_D}R$ by (1). Thus $(IR)_{t_R} \subseteq I_{t_D}R$. The reverse inclusion follows from Proposition 3.1(3); so $(IR)_{t_R} = I_{t_D}R$.

The statements (3) and (4) follow from Proposition 3.1. (1), (2) $\Rightarrow$ (5) is obvious. (1) $\Rightarrow$ (6) follows from Propositions 3.1 and 3.2. □

Corollary 3.4 becomes useful when for example:

(A) $R = D[X]$, where X is an indeterminate over D. Nishimura [35] proved that $(ID[X])^{-1} = I^{-1}D[X]$ for every $I \in F(D)$, and (1) of Corollary 3.4. See also Hedstrom and Houston [26], where (2) of Corollary 3.4 was proven for this case.

(B) $R = D + XL[X]$, where K is a subfield of a field L. The $K = L$ case was touched on in [12], where it was shown using direct methods that (1) of Corollary 3.4

holds. The $K \subsetneq L$ case was considered in [13]. But both of these fall under what came to be known as the "generalized $D + M$ construction" ([10]), which can be described as follows: Let T be an integral domain of the form $L + M$, where L is a field and M is a maximal ideal of T. Then let D be a subring of L, and let $R = D+M$. The first author and Rykaert [6] noted that $(IR)^{-1} = I^{-1}+M = I^{-1}R$ for every $I \in I(D)$. The special case when T is a valuation domain was studied by Bastida and Gilmer in [9]. It is interesting to note that in all of these cases, R is at least a faithfully flat extension of D.

A number of the known examples where Theorem 1.1 seems to be at work fall under the case where $D \subseteq R$ is an extension of integral domains with the property that $R :_L IR = (D :_K I)R$ for every $I \in f(D)$. Not all such extensions are flat. In the following corollary, we replace the $I \in F(D)$ hypothesis of Corollary 3.4 by $I \in f(D)$. Recall that an integral domain D is a *Prüfer v-multiplication domain* (PVMD) if the set of fractional v-ideals of finite type of D forms a group under v-multiplication.

Corollary 3.5. *Let R be an integral domain with quotient field L, and let D be a subring of R with quotient field K. If $R :_L IR = (D :_K I)R$ for every $I \in f(D)$, then the following statements hold.*

(1) $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ *for every* $I \in f(D)$.
(2) $(IR)_{t_R} = (I_{t_D}R)_{t_R}$ *for every* $I \in F(D)$.
(3) $(IR)_{v_R} = (I_{t_D}R)_{v_R}$ *for every* $I \in F(D)$.
(4) *If I is an integral t-ideal of R such that $I \cap D \neq (0)$, then $I \cap D$ is a t-ideal of D.*
(5) *If D has the additional property that I^{-1} is of finite type for every $I \in f(D)$ (e.g., if D is a PVMD), then the following statements hold.*
   (a) $(IR)_{v_R} = I_{v_D}R$ *for every* $I \in f(D)$.
   (b) $(IR)_{t_R} = I_{t_D}R$ *for every* $I \in F(D)$.
   (c) $(IR)_{v_R} = (I_{t_D}R)_{v_R}$ *for every* $I \in F(D)$.
   (d) *If J is an integral t-ideal of D, then JR is a t-ideal of R.*

The known cases that fall under the hypothesis of Corollary 3.5 are of the following types.

(i). When R is a ring of fractions of D. This case was studied in [40].

(ii). When R is a flat overring of D. This case was studied in [39], also see Fontana and Gabelli [18, Proposition 0.6]. But in each case, the authors were interested in proving (1) of Corollary 3.5.

(iii). When $R = D + XD_S[X]$, where S is a multiplicative subset of $D \setminus \{0\}$ and X is an indeterminate over D. In [41, Lemma 3.1], it was shown that the hypothesis of Corollary 3.5 holds in this case and only parts of (5) above were used. A construction that is closely related to $R = D + XD_S[X]$ is the construction $T = D + X\mathfrak{D}[X]$, where D is a subring of $\mathfrak{D}$, which was studied in [2] and has received considerable attention from a number of authors. Kabbaj and the first two authors in [5, Lemma 3.6] showed that $T = D + X\mathfrak{D}[X]$ is a flat D-module if and only if $\mathfrak{D}$ is a flat D-module; and so for $\mathfrak{D}$ a flat D-module, the hypothesis of Corollary 3.5 holds. They too used parts of (5) above in their work. This construction, which is customarily denoted by $A+XB[X]$, can do much more than $D + XD_S[X]$ can. A reader interested in this construction may want to check the references given in [5] and [33]. We have that the extension $D \subseteq D + X\mathfrak{D}[X]$ satisfies the hypothesis of Corollary 3.4 or that of Corollary 3.5 if and only if $D \subseteq \mathfrak{D}$ does. Indeed, let $I \in F(D)$. By [5, Lemma 2.1], we have $(IT)^{-1} = (I^{-1} \cap (I\mathfrak{D})^{-1}) + X(I\mathfrak{D})^{-1}[X] = I^{-1} + X(I\mathfrak{D})^{-1}[X]$. Hence $(IT)^{-1} = I^{-1}T$ if and only if $(I\mathfrak{D})^{-1} = I^{-1}\mathfrak{D}$.

(iv). In certain pullback constructions, see [18, Proposition 1.8].

The cases where Theorem 1.1 appears to be at work in full force seem to fall under the following category:

When $D \subseteq R$ is an extension of integral domains and R is an intersection of overrings R_α (i.e., $R \subseteq R_\alpha \subseteq L$) such that $D \subseteq R_\alpha$ satisfies the hypothesis of Corollary 3.5 for every α, i.e., $(IR_\alpha)^{-1} = I^{-1}R_\alpha$ for every $I \in f(D)$, where $I^{-1} = D :_K I$. This would happen when, for instance, every R_α is a flat D-module. In this case, we have $R{:_L}IR = (\bigcap R_\alpha) :_L IR = \bigcap (R_\alpha :_L IR_\alpha) = \bigcap I^{-1}R_\alpha = (I^{-1}R)^*$, where $*$ is the star operation induced by $\{R_\alpha\}$ on R ([1]). Indeed, as the v-operation is coarser than any other star operation and the extreme left expression in these equations is a v-ideal, we have the result. Consequently, if R is an overring of D such that R is an intersection of flat overrings of D, then Theorem 1.1 applies to the extension $D \subseteq R$.

The case $D \subseteq R$, where R is a generalized ring of fractions of D, is somewhat peculiar. The generalized ring of fractions is defined as follows. Let $\mathfrak{S}$ be a generalized multiplicative system, i.e., a multiplicative set generated by a nonempty set of nonzero (integral) ideals of D. Then $D_\mathfrak{S} = \{x \in K \mid xI \subseteq D \text{ for some } I \in \mathfrak{S}\}$ is a ring called the *generalized ring of fractions* with respect to $\mathfrak{S}$. There are two kinds of ideal extensions from D to $D_\mathfrak{S}$. Given an ideal J of D, we define $J_\mathfrak{S} = \{x \in K \mid xI \subseteq J \text{ for some } I \in \mathfrak{S}\}$. It is well known that $JD_\mathfrak{S} \subseteq J_\mathfrak{S}$. For further details, the reader may consult [7] and the references given there. For a more recent treatment of the topic, see [20]. In [36, Lemme 1], Querré stated that if $\mathfrak{S}$ is a generalized multiplicative system and A and B are ideals of D, then $(A{:_K}B)_\mathfrak{S} \subseteq A_\mathfrak{S}{:_K}B_\mathfrak{S}$ and if B is of finite type, then $(A :_K B)_\mathfrak{S} = A_\mathfrak{S} :_K B_\mathfrak{S} = A_\mathfrak{S} :_K BD_\mathfrak{S}$. This means that (i) $(B^{-1})_\mathfrak{S} = (B_\mathfrak{S})^{-1} = (BD_\mathfrak{S})^{-1}$ for every finitely generated ideal B of D. Kang [32, Lemma 3.4] extended this result to (ii) $(B_\mathfrak{S})_v = (BD_\mathfrak{S})_v = (B_{v\mathfrak{S}})_v = (B_vD_\mathfrak{S})_v$ when B is finitely generated and (iii) $(BD_\mathfrak{S})_t = (B_tD_\mathfrak{S})_t$ for every $B \in F(D)$, see also [22]. We conclude that $D \subseteq D_\mathfrak{S}$ is t-compatible. Now under certain conditions, $D_\mathfrak{S}$ is as an intersection of localizations of D and under these conditions, as we have already seen, we have $(B^{-1}D_\mathfrak{S})_v = (BD_\mathfrak{S})^{-1}$ for every $B \in f(D)$. This leads to the following question: Is it true that every extension $D \subseteq R$ of integral domains, where R is a generalized ring of fractions of D, satisfies the hypothesis (8) of Theorem 1.1? The answer below shows that this happens if and only if $\mathfrak{S}$ is a localizing system, where a *localizing system* is a nonempty family $\mathcal{F}$ of nonzero integral ideals of D satisfying:

(LS1) If $I \in \mathcal{F}$ and $J \in I(D)$ with $I \subseteq J$, then $J \in \mathcal{F}$;

(LS2) If $I \in \mathcal{F}$ and $J \in I(D)$ with $(J :_D iD) \in \mathcal{F}$ for every $i \in I$, then $J \in \mathcal{F}$.

A localizing system is a special kind of generalized multiplicative system of ideals. If S is a multiplicative set of D, then $\mathcal{F} = \{I \in I(D) \mid I \cap S \neq \phi\}$ is a localizing system such that $D_\mathcal{F} = D_S$. In particular, if $S = D \backslash P$, where P is a prime ideal of D, then $\mathcal{F}_P = \{I \in I(D) \mid I \nsubseteq P\}$ is a localizing system such that $D_{\mathcal{F}_P} = D_P$ and $J_{\mathcal{F}_P} = J_P$ for every $J \in F(D)$. More generally, if $\triangle \subseteq \mathrm{Spec}(D)$ (the set of prime ideals of D), then $\mathcal{F}(\triangle) = \cap\{\mathcal{F}_P \mid P \in \triangle\}$ is a localizing system of D and $D_{\mathcal{F}(\triangle)} = \cap_{P\in\triangle} D_P$.

In our dealings with localizing systems, we shall need the following three easy to establish facts: (1) $J_{\mathcal{F}} = D_{\mathcal{F}}$ for an ideal J of D if and only if $J \in \mathcal{F}$, (2) $(xI)_{\mathcal{F}} = xI_{\mathcal{F}}$ for every $0 \neq x \in K$ and for every ideal I of D, and (3) if E is a D-submodule of $D_{\mathcal{F}}$, then $E_{\mathcal{F}} \subseteq D_{\mathcal{F}}$ (see [19, Section 2]).

We say that a localizing system $\mathcal{F}$ is *v-complete* if for every family $\{I_k\}$ of divisorial ideals in $\mathcal{F}$ such that $\bigcap I_k \neq (0)$, we have $\bigcap I_k \in \mathcal{F}$. There exist localizing systems that are v-complete and there exist ones that are not; here is a simple example to establish that.

Example 3.6. Let V be a valuation domain and P a nonzero idempotent prime ideal of V. Then $\overline{\mathcal{F}}_P = \{I \in I(V) \mid I \supseteq P\}$ is a v-complete localizing system (cf. [20, Proposition 5.1.12]). For an example of a localizing system that is not v-complete, let x be a nonunit of V. Set $Q = \bigcap_{k\geq 1}(x^k V)$. Note that Q is a prime ideal of V and $Q \subsetneq x^k V$ for every integer $k \geq 1$ (cf. [23, Theorem 17.1]). The localizing system $\mathcal{F}_Q = \{I \in I(V) \mid I \supsetneq Q\}$ is not v-complete since the family $\{x^k V\}_{k=1}^{\infty}$ is in $\mathcal{F}_Q$, but $\bigcap_{k\geq 1}(x^k V) = Q \notin \mathcal{F}_Q$.

The following theorem characterizes overrings of an integral domain that are strongly t-compatible extensions.

Theorem 3.7. *Let R be an overring of an integral domain D. Then the following statements are equivalent.*

(1) $(IR)^{-1} = (I^{-1}R)_v$ *for every* $I \in f(D)$.
(2) $R = D_{\mathcal{F}}$ *for some localizing system* $\mathcal{F}$ *of* D.

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{F} = \{I \in I(D) \mid (IR)_v = R\}$. We first show that $\mathcal{F}$ is a localizing system. It is clear that $I, J \in \mathcal{F} \Rightarrow IJ \in \mathcal{F}$ and that $\mathcal{F}$ satisfies (LS1). For (LS2), let $I \in \mathcal{F}$ and $J \in I(D)$ such that $(J :_D iD) \in \mathcal{F}$ for every $i \in I$. Then $i(J :_D iD) \subseteq J$, and thus $i(J :_D iD)R \subseteq JR$. Since $((J :_D iD)R)_v = R$, we conclude that $iR \subseteq (JR)_v$ for every $i \in I$, and hence $IR \subseteq (JR)_v$. Since $I \in \mathcal{F}$, we have $(IR)_v = R$, which forces $(JR)_v = R$, and consequently $J \in \mathcal{F}$. Thus $\mathcal{F}$ is a localizing system.

We now show that $R = D_{\mathcal{F}}$. Let $x \in D_{\mathcal{F}}$; then $xI \subseteq D$ for some $I \in \mathcal{F}$. Thus $xIR \subseteq R$, and hence $x \in R$ since $(IR)_v = R$. Thus $D_{\mathcal{F}} \subseteq R$. For the reverse inclusion, let $x \in R$. We have $((x^{-1}D \cap D)R)_v = ((1,x)^{-1}R)_v = ((1,x)R)^{-1} = R$ by (1); so $x^{-1}D \cap D \in \mathcal{F}$. Since $x(x^{-1}D \cap D) \subseteq D$, we have $x \in D_{\mathcal{F}}$. Therefore $R = D_{\mathcal{F}}$.

(2) $\Rightarrow$ (1) Let $\mathcal{F}$ be a localizing system of D such that $R = D_{\mathcal{F}}$, and let $I \in f(D)$. Note that $I^{-1}R \subseteq (IR)^{-1}$ always holds, and thus $(I^{-1}R)_v \subseteq (IR)^{-1}$. Recall that $(I^{-1}R)_v = \bigcap\{yR \mid y \in K \text{ and } I^{-1}R \subseteq yR\}$. So for the reverse inclusion, we only need to show that $(IR)^{-1} \subseteq yR$ for every $y \in K$ such that $I^{-1}R \subseteq yR$. Let $x \in (IR)^{-1}$. Since, according to Querré [36, Lemme 1], $(IR)^{-1} = (ID_{\mathcal{F}})^{-1} = (D : I)_{\mathcal{F}}$ because I is finitely generated, we have $x \in (D : I)_{\mathcal{F}}$, which means $xJ \subseteq D : I = I^{-1}$ for some $J \in \mathcal{F}$. This gives $xJR \subseteq I^{-1}R \subseteq yR$ for every

$yR \supseteq I^{-1}R$. Thus $xJR \subseteq yR$ or $xy^{-1}JR \subseteq R$, which gives $xy^{-1}J \subseteq R = D_{\mathcal{F}}$, and hence $(xy^{-1}J)_{\mathcal{F}} \subseteq D_{\mathcal{F}} = R$. Since $(xy^{-1}J)_{\mathcal{F}} = xy^{-1}J_{\mathcal{F}} = xy^{-1}D_{\mathcal{F}} = xy^{-1}R$, we have $xy^{-1}R \subseteq R$, and thus $xR \subseteq yR$, which forces $x \in yR$ for every y such that $yR \supseteq I^{-1}R$. Hence $x \in (IR)^{-1}$ implies $x \in \bigcap\{yR \mid y \in K \text{ and } I^{-1}R \subseteq yR\} = (I^{-1}R)_v$, which establishes the reverse inclusion, and thus the equality. □

Corollary 3.8. *Let $\mathcal{F}$ be a localizing system of an integral domain D, and let $R = D_{\mathcal{F}}$. Then the following statements hold.*

1. $I_{v_D}R \subseteq (IR)_{v_R}$ *for every* $I \in f(D)$.
2. $(IR)_{v_R} = (I_{v_D}R)_{v_R}$ *for every* $I \in f(D)$.
3. $I_{t_D}R \subseteq (IR)_{t_R}$ *for every* $I \in F(D)$.
4. $(IR)_{t_R} = (I_{t_D}R)_{t_R}$ *for every* $I \in F(D)$.
5. $(IR)_{v_R} = (I_{t_D}R)_{v_R}$ *for every* $I \in F(D)$.
6. *If I is an integral t-ideal of R, then $I \cap D$ is a t-ideal of D.*
7. *If I is a nonzero principal fractional ideal of R, then $I \cap D$ is a t-ideal of D.*

Proof. By Theorem 3.7, $R = D_{\mathcal{F}}$ satisfies the hypothesis (8) of Theorem 1.1. Thus statements (1)-(7) all hold. (For (6) and (7), note that $I \cap D$ is nonzero when I is nonzero since R is an overring of D.) □

Replacing $I \in f(D)$ with $I \in F(D)$ in Theorem 3.7, we have the following result.

Theorem 3.9. *Let R be an overring of an integral domain D. Then the following statements are equivalent.*

1. $(IR)^{-1} = (I^{-1}R)_v$ *for every* $I \in F(D)$.
2. $R = D_{\mathcal{F}}$ *for some v-complete localizing system $\mathcal{F}$ of D.*
3. $((\bigcap I_k)R)_v = (\bigcap(I_kR))_v = \bigcap(I_kR)_v$ *for every family $\{I_k\}$ of fractional divisorial ideals of D such that* $\bigcap I_k \neq (0)$.

Proof. (1) ⇒ (3) By [30, Lemma 1.1], we have $\bigcap I_k = (\sum I_k^{-1})^{-1}$. Thus $((\bigcap I_k)R)_v = ((D : \sum I_k^{-1})R)_v = R : (\sum I_k^{-1})R = R : \sum(I_k^{-1}R)_v = R : \sum(I_kR)^{-1} = \bigcap(I_kR)_v$. The middle equality follows once we observe that $((\bigcap I_k)R)_v \subseteq (\bigcap(I_kR))_v \subseteq \bigcap(I_kR)_v$.

(3) ⇒ (1) Let $I \in I(D)$. We have $R : IR = \bigcap\{i^{-1}R \mid 0 \neq i \in I\}$. For every $0 \neq i \in I$, set $I_i = i^{-1}D$. Consider the family of divisorial ideals $\{I_i\}_{i \in I\setminus\{0\}}$. Note that since $1 \in I_i$ for every i, we have $\bigcap I_i \neq (0)$ and that $\bigcap I_i = D : I$. Thus $R : IR = \bigcap(I_iR) = (\bigcap(I_iR))_v = ((\bigcap I_i)R)_v = ((D : I)R)_v$.

(1) ⇒ (2) Let $\mathcal{F} = \{I \in I(D) \mid (IR)_v = R\}$. As in the proof of Theorem 3.7, $\mathcal{F}$ is a localizing system and $R = D_{\mathcal{F}}$. We next show that $\mathcal{F}$ is v-complete. Let $\{I_k\}$ be a family of divisorial ideals in $\mathcal{F}$ such that $\bigcap I_k \neq (0)$. By (1) ⇔ (3), we get $((\bigcap I_k)R)_v = \bigcap(I_kR)_v = R$. Hence $\bigcap I_k \in \mathcal{F}$.

(2) ⇒ (1) Assume that $R = D_{\mathcal{F}}$ for some v-complete localizing system $\mathcal{F}$ of D. Let $I \in I(D)$. We first show that $(ID_{\mathcal{F}})^{-1} \subseteq (I^{-1})_{\mathcal{F}}$. Let $x \in (ID_{\mathcal{F}})^{-1}$. Then $xI \subseteq D_{\mathcal{F}}$, which means that for every $i \in I$, there is a $J_i \in \mathcal{F}$ such that $xiJ_i \subseteq D$. Write $x = ab^{-1}$ with $a, b \in D$ and $b \neq 0$, and set $H_i = (J_i + bD)_v$. Then $\{H_i\}_{i\neq 0}$ is a family of divisorial ideals in $\mathcal{F}$ such that $\bigcap_{i\neq 0} H_i \neq (0)$ and $xiH_i \subseteq D$ for every $i \in I\setminus\{0\}$. Hence $x(\bigcap_{i\neq 0} H_i) \subseteq \bigcap_{i\neq 0} i^{-1}D = D : I$. But $\bigcap_{i\neq 0} H_i \in \mathcal{F}$ because $\mathcal{F}$ is v-complete. Thus $x \in (I^{-1})_{\mathcal{F}}$, and hence $(ID_{\mathcal{F}})^{-1} \subseteq (I^{-1})_{\mathcal{F}}$.

We claim that $(HD_{\mathcal{F}})_v = D_{\mathcal{F}}$ for every $H \in \mathcal{F}$. Indeed, by the above result, we have $(HD_{\mathcal{F}})^{-1} \subseteq (H^{-1})_{\mathcal{F}}$. Moreover, according to [36], $(H^{-1})_{\mathcal{F}} = (D{:}H)_{\mathcal{F}} \subseteq D_{\mathcal{F}}: H_{\mathcal{F}} = D_{\mathcal{F}}$. Thus $D_{\mathcal{F}} \subseteq (HD_{\mathcal{F}})^{-1} \subseteq (H^{-1})_{\mathcal{F}} \subseteq D_{\mathcal{F}}$, which implies $(HD_{\mathcal{F}})_v = D_{\mathcal{F}}$. Now let $x \in (ID_{\mathcal{F}})^{-1}$. Since $(ID_{\mathcal{F}})^{-1} \subseteq (I^{-1})_{\mathcal{F}}$, there is an $H \in \mathcal{F}$ such that $xH \subseteq I^{-1}$, or $xHD_{\mathcal{F}} \subseteq I^{-1}D_{\mathcal{F}}$, or $x(HD_{\mathcal{F}})_v \subseteq (I^{-1}D_{\mathcal{F}})_v$. Hence $x \in (I^{-1}D_{\mathcal{F}})_v$, and thus $(ID_{\mathcal{F}})^{-1} \subseteq (I^{-1}D_{\mathcal{F}})_v$. The reverse inclusion is obvious. □

The following example shows that there exist proper extensions of integral domains that satisfy Theorem 3.7, but not Theorem 3.9.

Example 3.10. Let (V, M) be a rank-two valuation domain such that V has no nonzero idempotent prime ideals (i.e., V has value group $\mathbb{Z} \oplus_L \mathbb{Z}$). For P the height-one prime ideal of V, let $R = V_{\mathcal{F}_P} = V_P$. Then the extension $V \subseteq R$ clearly satisfies the equivalent conditions of Theorem 3.7. Let $x \in M \backslash P$. Then $P = \bigcap_{k \geq 1}(x^k V)$. As we have seen in Example 3.6, the localizing system $\mathcal{F}_P$ is not v-complete. Suppose that there is a v-complete localizing system $\mathcal{F}$ such that $R = V_{\mathcal{F}}$. Then $\mathcal{F} = \mathcal{F}_Q$ for some prime ideal Q by [20, Proposition 5.1.12]. Necessarily $Q = M$. Thus $\mathcal{F} = \mathcal{F}_M$; so $R = V_M = V$, which is impossible.

Remark 3.11. (1) An interesting particular case of Theorem 3.9 is when $R = D_P$ for some prime ideal P of D. This case was studied in [17]. More precisely, we have the following equivalences by [17, Lemma 2.1]:

(i) $(ID_P)^{-1} = I^{-1}D_P$ for every $I \in F(D)$.
(ii) $\mathcal{F}_P$ is a v-complete localizing system.
(iii) For every family $\{I_\alpha\}$ of integral divisorial ideals of D such that $\bigcap I_\alpha \neq (0)$, $\bigcap I_\alpha \subseteq P \Rightarrow I_\alpha \subseteq P$ for some α.
(iv) $(\bigcap I_\alpha)D_P = \bigcap(I_\alpha D_P)$ for every family $\{I_\alpha\}$ of fractional divisorial ideals of D such that $\bigcap I_\alpha \neq (0)$.

(2) Besides Theorem 3.9, we have the following characterization of a v-complete localizing system. Let $\mathcal{F}$ be a localizing system of D; then the following statements are equivalent.

(i) $\mathcal{F}$ is a v-complete localizing system.
(ii) $(\bigcap I_\alpha)_{\mathcal{F}} = \bigcap(I_\alpha)_{\mathcal{F}}$ for every nonempty family $\{I_\alpha\}$ of divisorial fractional ideals of D such that $\bigcap I_\alpha \neq (0)$.

Note that this equivalence generalizes (ii) $\Leftrightarrow$ (iv) of (1). For (i) $\Rightarrow$ (ii), let $x \in \bigcap(I_\alpha)_{\mathcal{F}}$. Then for each α, there exists a $J_\alpha \in \mathcal{F}$ such that $xJ_\alpha \subseteq I_\alpha$. By an argument similar to the one used in the proof of (2) $\Rightarrow$ (1) of Theorem 3.9, we can assume that $\{J_\alpha\}$ is a family of divisorial ideals such that $\bigcap J_\alpha \neq (0)$. Thus $x(\bigcap J_\alpha) \subseteq \bigcap I_\alpha$ implies that $x \in (\bigcap I_\alpha)_{\mathcal{F}}$ since $\mathcal{F}$ is v-complete. Hence $\bigcap(I_\alpha)_{\mathcal{F}} \subseteq (\bigcap I_\alpha)_{\mathcal{F}}$. The other inclusion is clear. For the converse, let $\{I_\alpha\}$ be a family of divisorial ideals in $\mathcal{F}$ such that $\bigcap I_\alpha \neq (0)$. Then $(\bigcap I_\alpha)_{\mathcal{F}} = \bigcap(I_\alpha)_{\mathcal{F}} = D_{\mathcal{F}}$. Hence $\bigcap I_\alpha \in \mathcal{F}$.

The next example shows that an integral domain D may have an overring R which is a t-compatible (or v-compatible) extension of D that is not a generalized ring of fractions of D.

Example 3.12. Let $k \subset K$ be a proper extension of fields. Let $V = K[[X]] = K + M$ with $M = XV$, and let $D = k + M$. Let $I \in F(D)$. By [9, Theorem

4.3], one can easily check that $I_v \subseteq (IV)_v$, that is, $D \subseteq V$ is v-compatible, and hence t-compatible. We next show that V is not a generalized ring of fractions of D. Suppose that $V = D_{\mathcal{F}}$ for some generalized multiplicative system $\mathcal{F}$ of D. Let $I \in \mathcal{F} \setminus \{D\}$. Then $I \subseteq M$ implies $I^2 \subseteq M^2$; so $D : M^2 \subseteq D : I^2 \subseteq D_{\mathcal{F}} = V$. Thus $D : M^2 = V$. But $D : M^2 = D : X^2V = X^{-2}(D : V) = X^{-2}M$; so $M = X^2V$, a contradiction.

4. Integral domains whose overrings are t-compatible extensions

Following Richman's characterization of Prüfer domains by means of their overrings [38], various conditions on the set of overrings of a given integral domain were considered in order to study integral domains with "Prüfer-like" behavior.

Recall that a *QR-domain* (resp., *QQR-domain*) is an integral domain D such that every overring of D is a ring of fractions (resp., an intersection of rings of fractions) of D (cf. [24, 25]). By [38, Theorem 4], a QR-domain is a Prüfer domain, but the converse is not true in general (cf. [14, 25]). On the other hand, since flat overrings are intersection of localizations ([38]), it is obvious that a Prüfer domain is a QQR-domain. The converse is not true since a QQR-domain is not necessarily integrally closed ([24, Example 4.1]). However, the integral closure of a QQR-domain is a Prüfer domain ([24]).

More generally, a *GQR-domain* (resp., *$\mathcal{F}$QR-domain*) is an integral domain D whose overrings are generalized rings of fractions with respect to multiplicative sets of ideals (resp., localizing systems) of D (cf. [21, 27]). It is obvious that

QR-domain $\Rightarrow$ QQR-domain $\Rightarrow$ $\mathcal{F}$QR-domain $\Rightarrow$ GQR-domain.

Since a generalized quotient ring of an integrally closed domain is integrally closed ([20, Lemma 5.1.14]), an integrally closed GQR-domain is a Prüfer domain. W. Heinzer [27] conjectured that the integral closure of a GQR-domain is a Prüfer domain. In [21], the authors proved this conjecture for $\mathcal{F}$QR-domains.

In the following, we extend the above results to integral domains whose overrings are all t-compatible extensions. An integral domain with this property will be called a *t-compatible domain*. Also, we say that an integral domain is *strongly t-compatible* if all of its overrings are strongly t-compatible extensions. So a strongly t-compatible domain is t-compatible. Note that QR-domains are strongly t-compatible and QQR, $\mathcal{F}$QR, and GQR-domains are all t-compatible. By Theorem 3.7, strongly t-compatible domains coincide with $\mathcal{F}$QR-domains.

If D is a PVMD, the notion of t-compatible extension coincides with that of strongly t-compatible extension.

Proposition 4.1. *Let $D \subseteq R$ be an extension of integral domains. If D is a PVMD, then the following statements are equivalent.*

(1) *$D \subseteq R$ is strongly t-compatible.*
(2) *$D \subseteq R$ is t-compatible.*
(3) *$D \subseteq R$ is t-linked.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) We need to show that $(IR)^{-1} = (I^{-1}R)_v$ for every $I \in f(D)$. Let $x \in (IR)^{-1}$. Then $xIR \subseteq R$ implies $xII^{-1}R \subseteq I^{-1}R$. Since D is a PVMD, we

have $(II^{-1})_t = D$, and thus $(II^{-1}R)_t = R$ by t-linkedness. Hence $x \in (I^{-1}R)_t \subseteq (I^{-1}R)_v$, and thus $(IR)^{-1} \subseteq (I^{-1}R)_v$. The other inclusion is clear; so $(IR)^{-1} = (I^{-1}R)_v$. □

Note that from the previous sections, for an extension of integral domains the following implications can not be reversed in general: strongly t-compatible ⇒ t-compatible ⇒ t-linked.

We next study the integrally closed t-compatible domains. Let $f \in K[X]$. We denote by $C_D(f)$ the *content* of f, i.e., the fractional ideal of D generated by the coefficients of f. We will need the following characterization of integrally closed domains due to Querré [37]: an integral domain D is integrally closed if and only if $C_D(fg)_v = (C_D(f)C_D(g))_v$ for every $0 \neq f, g \in K[X]$.

Theorem 4.2. *Let $D \subseteq R$ be a t-compatible extension of integral domains with R an overring of D. If D is integrally closed, then R is integrally closed.*

Proof. We prove that if the above formula on the content of two polynomials is satisfied for D, then it is also satisfied for R. Let $0 \neq f, g \in K[X]$. We have $C_R(f) = C_D(f)R$ and $C_R(g) = C_D(g)R$. By t-compatibility, we have $C_D(fg)_v \subseteq (C_D(fg)R)_v = C_R(fg)_v$. Since $C_D(fg)_v = (C_D(f)C_D(g))_v$ by assumption, it follows that $C_D(f)C_D(g) \subseteq C_R(fg)_v$. Thus $C_R(f)C_R(g) \subseteq C_R(fg)_v$, and hence $(C_R(f)C_R(g))_v \subseteq C_R(fg)_v$. The reverse inclusion is clear; so we have $C_R(fg)_v = (C_R(f)C_R(g))_v$. □

Corollary 4.3. *Let D be an integrally closed domain. Then the following statements are equivalent.*

(1) *D is a Prüfer domain.*
(2) *$(IR)^{-1} = I^{-1}R$ for every overring R of D and $I \in f(D)$.*
(3) *D is a strongly t-compatible domain.*
(4) *D is a t-compatible domain.*
(5) *D is a QQR-domain.*
(6) *D is a $\mathcal{F}$QR-domain.*
(7) *D is a GQR-domain.*

Proof. The fact that statements (5)-(7) are equivalent to D being a Prüfer domain is well known, as it was mentioned above. Since every overring of a Prüfer domain is flat, we have (1)⇒(2); (2) ⇒ (3) is clear; and (3) ⇒ (4) follows from Theorem 1.1. Finally, (4) ⇒ (1) follows from Theorem 4.2 and [14]. □

Remark 4.4. Since strongly t-compatible domains coincide with $\mathcal{F}$QR-domains by [21], the integral closure of a strongly t-compatible domain is a Prüfer domain. We can ask the same question about t-compatible domains, but this question is still open in the case of GQR-domains ([27]).

Analogously, we say that an integral domain D is *v-compatible* if every overring of D is a v-compatible extension. A v-compatible domain is t-compatible. The converse is not true since a Prüfer domain is t-compatible, but not v-compatible in general (see below).

Corollary 4.5. *An integrally closed v-compatible domain is a Prüfer domain.*

The following example shows that a Prüfer domain need not be v-compatible.

Example 4.6. ([28, Example 2.6]) Let D be a two-dimensional Prüfer domain with maximal ideals M_1, M_2 and P the height-one prime ideal contained in $M_1 \cap M_2$ with the assumption that D_P is a DVR. Recall that P is a divisorial ideal and $P^{-1} = (P : P) = D_P$, see [31]. Let $I = PD_{M_1} \cap xD_{M_2}$, where $PD_P = xD_P$ for some $x \in P$. By [28, Example 2.6], we have $I^{-1} = P^{-1} = D_P$ and $(I : I) \subsetneq D_P$. In particular, $I_v = P_v = P$. Assume that $I_v \subseteq (ID_{M_2})_v$. But $ID_{M_2} = xD_{M_2}$; so $P = I_v \subseteq (ID_{M_2})_v = xD_{M_2}$. Thus $P \subseteq I$; so $I = P$, which is impossible since $(I : I) \subsetneq (P : P)$. Hence the extension $D \subseteq D_{M_2}$ is not v-compatible.

Remark 4.7. As we have seen above, a Prüfer domain need not be v-compatible. What about valuation domains? In this case, the answer is positive. Indeed, let V be a valuation domain and W be a proper overring of V. Then $W = V_P$ for some non-maximal prime ideal P of V. Let $I \in F(V)$ and $x \in (IV_P)^{-1}$. Then $xI \subseteq V_P$ implies $xIP \subseteq PV_P = P$. Taking the v-closure, we get $xI_vP_v \subseteq P_v$. Thus $xI_v \subseteq P_v : P_v = P^{-1} = V_P$ (cf. [31]), and hence $x \in (I_vV_P)^{-1}$. Thus $(I_vW)_v \subseteq (IW)_v$, and hence $(I_vW)_v = (IW)_v$. Therefore $V \subseteq W$ is v-compatible.

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