t-INVERTIBILITY AND BAZZONI-LIKE STATEMENTS

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Abstract. We show that if D is an integral domain such that every nonzero locally principal ideal of D is invertible then every invertible integral ideal of D is contained in at most a finite number of mutually comaximal invertible ideals. We use this result to provide a direct verification of Bazzoni's conjecture: A Prüfer domain D such that every nonzero locally principal ideal of D is invertible is of finite character. We also discuss some, star-operation- theoretic variants of the above mentioned conjecture.

Dedicated to Paulo Ribenboim, his work serves as a beacon for the likes of me.

Bazzoni in [2] and in [3] put forward the conjecture: If D is a Prüfer domain such that every locally principal ideal of D is invertible then D is of finite character. (D is Prüfer if every finitely generated nonzero ideal of D is invertible and D is of finite character if every nonzero nonunit of D belongs to only finitely many maximal ideals of D.) This conjecture was resolved by Holland, Martinez, McGovern, and Tesemma in [10] and later stated and proved Bazzoni's conjecture for the so called r-Prüfer monoids, which in the domain case are PVMD's and include Prüfer domains, by Halter-Koch [8]. The aim of this note is to introduce a device that not only verifies Bazzoni's conjecture for all the above cases but also allows us to prove Bazzoni like statements in more general domains. Our plan is to prove a general theorem, to almost verify the Bazzoni Conjecture, as part of introduction/motivation. We then introduce the readers to star operations and verify Bazzoni Conjecture for the PVMD's and finally produce some Bazzoni-like statements for domains that are not PVMD's.

Theorem 1. *Let D be an integral domain. If D contains a nonzero element x such that x is contained in infinitely many proper mutually comaximal invertible ideals then D contains an ideal that is locally principal yet not invertible. Equivalently if D is such that every locally principal ideal of D is invertible then each proper principal ideal of D is contained in at most a finite number of proper mutually co-maximal invertible ideals of D.*

We shall refer, in the sequel, to a known result of Griffin [6] that shows that the equivalently part of the above theorem is equivalent for a Prüfer domain D to be of finite character.

Proof. Let $S = \{A_i\}_{i \in N}$ be a collection of proper invertible ideals of D such that $0 \neq x \in A_i$ and $A_i + A_j = D$ for $i \neq j$. Since the members of S are mutually comaximal we have for each $n \in N$, $A_1 \cap A_2 \cap ... \cap A_n = A_1 A_2 ... A_n$. So

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$x \in A_1A_2...A_n$ for each $n \in N$. Thus $(A_1A_2...A_n)^{-1}x \subseteq D$ for each $n \in N$. Next as A_i are all proper and invertible $(A_1A_2...A_nA_{n+1})^{-1} \supsetneq (A_1A_2...A_n)^{-1}$ and so $(A_1A_2...A_n)^{-1}x \subsetneq (A_1A_2...A_nA_{n+1})^{-1}x$ for each $n \in N$. This gives us a strictly ascending sequence of ideals $\{(A_1A_2...A_n)^{-1}x\}$. Consider the ideal $F = \sum_{n=1}^{\infty}(A_1A_2...A_n)^{-1}x$ and note that $F = \bigcup_{n=1}^{\infty}(A_1A_2...A_n)^{-1}x$. Also note that F cannot be finitely generated and hence not invertible because F is a union of an infinite strictly ascending chain. Now to see that F is locally principal we note that no maximal ideal can contain any pair of distinct members of S. So if M is a maximal ideal of D such that no member of S is contained in M then $FD_M = xD_M$ and if F contains A_i for some i then for $i = 1$ we have $FD_M = A_1^{-1}xD_M$ and for $i > 1$ we have $FD_M = xD_M + A_i^{-1}xD_M = A_i^{-1}xD_M$. Thus in each case we have FD_M principal and this completes the proof. The "equivalently" part is simply a contrapositive of the first part. □

Corollary 1. *In a Noetherian domain every nonzero nonunit can belong to only a finite number of mutually comaximal proper invertible ideals.*

Now to see that for a Prüfer domain the above theorem delivers the goods and to prepare for the more general results we introduce below the notion of star operations. Most of the information given below can be found in [14] and [5, sections 32, 34]. Let D denote an integral domain with quotient field K and let $F(D)$ be the set of nonzero fractional ideals of D. A star operation $*$ on D is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(c) $(A^*)^* = A^*$.

For $A, B \in F(D)$ we define $*$-*multiplication* by $(AB)^* = (A^*B)^* = (A^*B^*)^*$. A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ where B is a finitely generated fractional ideal. A star operation $*$ is said to be of *finite character* if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. For $A \in F(D)$ define $A^{-1} = \{x \in K \mid xA \subseteq D\}$ and call $A \in F(D)$ $*$-*invertible* if $(AA^{-1})^* = D$. Clearly every invertible ideal is $*$-invertible for every star operation $*$. If $*$ is of finite character and A is $*$-invertible, then A^* is of finite type. The most well known examples of star operations are: the v-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the t-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$. Given two star operations $*_1, *_2$ we say that $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in F(D)$. Note that $*_1 \leq *_2$ if and only if $(A^{*_1})^{*_2} = (A^{*_2})^{*_1} = A^{*_2}$. By definition t is of finite character, $t \leq v$ while $\rho \leq t$ for every star operation ρ of finite character. If $*$ is a star operation of finite character then using Zorn's Lemma we can show that an integral ideal maximal w.r.t. being a star ideal is a prime ideal and that every integral $*$-ideal is contained in a maximal $*$-ideal. Let us denote the set of all maximal $*$-ideals by $* - max(D)$. It can also be easily established that for a star operation $*$ of finite character on D we have $D = \bigcap_{M \in * - \max(D)} D_M$. A v-ideal A of finite type is t-invertible if and only if A is t-locally principal i.e. for every $M \in t - \max(D)$ we have AD_M principal. An integral domain D is called a Prüfer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible. According to Griffin [6, Theorem 5] D

is a PVMD if and only if D_M is a valuation domain for each $M \in t - \max(D)$. The set $Inv_t(D) = \{A \in F(D) : A$ is a t-invertible t-ideal$\}$ is obviously a group under t-multiplication. If we define an order as $A \leq B$ if and only if $A \supseteq B$ then $< Inv_t(D), \leq, \times_t >$ is a directed group [14, Corollar 1.3]. Griffin [6, page 717] with reference to Jaffard [11, page 55] observes that for a PVMD $D < Inv_t(D)$, $\leq$, $\times_t >$ is a lattice ordered group, see [14, Proposition 2.4] for a direct proof and note that for $A, B \in Inv_t(D)$, $\sup(A, B) = A \cap B$ and $inf(A, B) = (A, B)_t$. Also Griffin proves in [6, Theorem 7] that every nonzero nonunit of a PVMD belongs to at most a finite number of maximal t-ideals if and only if $Inv_t(D)$ satisfies Conrad's F-condition: Every positive element is greater than only a finite number of mutually disjoint positive elements. Now as D is the identity of $< Inv_t(D), \leq, \times_t >$, by the definition of order $A \geq D$ implies that $A \subseteq D$ so positive elements of $< Inv_t(D)$, $\leq$, $\times_t >$ are precisely the integral t-invertible t-ideals. Moreover since in a p.o. group G two positive elements are disjoint if $\inf(A, B) =$ identity of G, two integral ideals A, B in $Inv_t(D)$ are disjoint if $(A, B)_t = D$ i.e. if A, B are t-comaximal and Griffin's result translates to the following result.

Proposition 1. *Every nonzero nonunit of a PVMD belongs to only a finite number of maximal t-ideals if and only if every integral t-invertible t-ideal of D is contained in at most a finite number of mutually t-comaximal t-invertible t-ideals.*

Griffin [6] called the PVMD's of Proposition 1, the rings of Krull type. Let us generally call a domain D of finite t-character if every nonzero nonunit of D belongs to at most a finite number of maximal t-ideals. Note that every integral t-invertible t-ideal belonging to only a finite number of maximal t-ideals is equivalent to every integral principal ideal belonging to only a finite number of maximal t-ideals.

Now the important observation, in a Prüfer domain every finitely generated $I \in F(D)$ is invertible and so is t-invertible. Thus a Prüfer domain is a PVMD. Also because for every finitely generated $I \in F(D)$, for D Prüfer, we have $I = (I^{-1})^{-1} = I_v$ and so every finitely generated ideal of a Prüfer domain is a v-ideal. From this we can also draw the conclusion that in a Prüfer domain every nonzero ideal is a t-ideal. In fact a PVMD D is a Prüfer domain if and only if every maximal ideal of D is a t-ideal [12, Proposition 4.4 (3)(b)] and so "t-comaximal" translates to comaximal. Consequently Proposition 1 translates to the following result.

Proposition 2. *A Prüfer domain D is a ring of finite character if and only if each invertible integral ideal of D is contained in at most a finite number of mutually comaximal invertible ideals.*

Proposition 3. *(Bazzoni's Theorem) A Prüfer domain D such that every locally principal ideal of D is invertible is of finite character.*

Proof. By Theorem 1, every proper principal ideal of D is contained in at most a finite number of mutually comaximal integral invertible ideals. This means that every finitely generated nonzero integral ideal of D is contained in at most a finite number of mutually comaximal integral invertible ideals of D. Now by Proposition 2 we have the result. □

Now for the general device we need to prepare a little. For a domain D the function $A \mapsto A_w = \bigcap_{M \in t-\max(D)} AD_M$ is also a star operation of finite character and so $(A_w)_t = A_t$. Let us recall also that if $A_1, A_2, ..., A_n$ are $*$-ideals, for a star operation $*$, then $(A_1 \cap A_2 \cap ... \cap A_n)^* = A_1 \cap A_2 \cap ... \cap A_n$.

Lemma 1. *If $A_1, A_2, ..., A_n$ are mutually t-comaximal t-ideals then $A_1 \cap A_2 \cap ... \cap A_n = (A_1A_2...A_n)_t$. Thus if $x \in A_i$ for every $i = 1, ..., n$ then $x \in (A_1A_2...A_n)_t$.*

Proof. Since A_i are mutually t-comaximal, they do not share a maximal t-ideal. Now if M is a maximal t-ideal that does not contain any of A_i, for $i = 1, ..., n$, then $D_M = (A_1 \cap A_2 \cap ... \cap A_n)D_M = (A_1A_2...A_n)D_M$. If on the other hand M is a maximal t-ideal that contains at least one and hence exactly one of them, say A_i then $A_iD_M = (A_1 \cap A_2 \cap ... \cap A_n)D_M = (A_1A_2...A_n)D_M$. Thus $(A_1 \cap A_2 \cap ... \cap A_n)_w = (A_1A_2...A_n)_w$, but this means $(A_1 \cap A_2 \cap ... \cap A_n)_t = (A_1A_2...A_n)_t$. But as A_i are t-ideals we have $A_1 \cap A_2 \cap ... \cap A_n = (A_1A_2...A_n)_t$. □

Lemma 2. *Let $A_1, A_2, ..., A_n, A_{n+1}$ be proper mutually t-comaximal t-invertible t-ideals such that $x \in A_i \backslash \{0\}$ for every $i = 1, ..., n$. Then $(\prod_{i=1}^{i=n} A_i^{-1})_t x \subsetneq (\prod_{i=1}^{i=n+1} A_i^{-1})_t x \subseteq D$. (The first proper inclusion holds for any $x \neq 0$, as the referee has rightly pointed out.)*

Proof. Since $A_{n+1}^{-1} \supseteq D$ we have $A_{n+1}^{-1}(\prod_{i=1}^{i=n} A_i^{-1})_t x \supseteq (\prod_{i=1}^{i=n} A_i^{-1})_t x$, for any $x \in D \backslash \{0\}$.Applying the t-operation on both sides we have $(\prod_{i=1}^{i=n} A_i^{-1})_t x \subseteq (\prod_{i=1}^{i=n+1} A_i^{-1})_t x$. To establish proper inclusion, set $(\prod_{i=1}^{i=n} A_i^{-1})_t x = (\prod_{i=1}^{i=n+1} A_i^{-1})_t x$, multiply both sides by $\frac{1}{x}\prod_{i=1}^{i=n} A_i$ and apply the t-operation on both sides to get $D = A_{n+1}^{-1}$ which contradicts the fact that A_{n+1} is a proper t-invertible t-ideal whence $(\prod_{i=1}^{i=n} A_i^{-1})_t x \subsetneq (\prod_{i=1}^{i=n+1} A_i^{-1})_t x$ the other inclusion follows from the fact that $x \in A_1 \cap A_2 \cap ... \cap A_{n+1} = (A_1 \cap A_2 \cap ... \cap A_{n+1})_t$ (because A_i are t-ideals) $= (A_1A_2...A_{n+1})_t$. Now multiplying both sides of $x \in (A_1A_2...A_{n+1})_t$ by $\prod_{i=1}^{i=n+1} A_i^{-1}$ and applying the t-operation, we get the inclusion. □

Proposition 4. *Let D be a domain that contains an integral t-invertible t-ideal A such that A is contained in an infinite number of mutually t-comaximal t-invertible t-ideals. Then in D there is a t-ideal F such that F is t-locally principal yet not a t-ideal of finite type and hence not t-invertible. Equivalently if every t-locally principal ideal of D is t-invertible then every integral t-invertible t-ideal of D is contained in at most a finite number of mutually t-comaximal t-invertible t-ideals of D.*

Proof. Suppose that an integral t-invertible t-ideal A of D is contained in an infinite set $\{H_i\}_{i \in N}$ with $(H_i + H_j)_t = D$ if $i \neq j$, of t-invertible t-ideals of D. Let $x \in A \backslash \{0\}$

and consider the sequence of ideals $H_1^{-1}x$, $(H_1^{-1}H_2^{-1})_t x$, ... $(\prod_{i=1}^{i=n} H_i^{-1})_t x$ ... . From Lemmata 1, 2, for each n, $(\prod_{i=1}^{i=n} H_i^{-1})_t x \subseteq D$, because H_i are mutually t-comaximal and for the same reasons $(\prod_{i=1}^{i=n} H_i^{-1})_t x \subsetneq (\prod_{i=1}^{i=n+1} H_i^{-1})_t x$.Consider the ideal $F = \sum_{n=1}^{\infty} (\prod_{i=1}^{i=n} H_i^{-1})_t x = \bigcup_{n=1}^{\infty} (\prod_{i=1}^{i=n} H_i^{-1})_t x$. Since $(\prod_{i=1}^{i=n} H_i^{-1})_t x \subset (\prod_{i=1}^{i=n+1} H_i^{-1})_t x$, for each n, F is an ascending union of t-ideals and hence is a t-ideal. Now F cannot be a t-ideal of finite type, because if say $F = (x_1, x_2, ..., x_r)_t$ then for some m we have $F \subseteq (\prod_{i=1}^{i=m} H_i^{-1})_t x$ and hence $F = (\prod_{i=1}^{i=m} H_i^{-1})_t x$ while $(\prod_{i=1}^{i=m} H_i^{-1})_t x \subsetneq (\prod_{i=1}^{i=m+1} H_i^{-1})_t x \subseteq F$ a contradiction. We now show that for each maximal t-ideal M, FD_M is principal. For this note that since H_i are mutually t-comaximal no maximal t-ideal contains two of them. Thus if M contains none of the H_i then $FD_M = xD_M$ and if $M = M_i$ for some $i \in N$ then $FD_{M_i} = H_i^{-1}x\ D_{M_i} = x(H_i D_{Mi})^{-1}$ which is again principal because $H_i D_{Mi}$ is principal. The equivalently part is just the contrapositive. □

Proposition 5. *A PVMD D is of finite t-character if and only if every t-locally principal t-ideal of D is t-invertible.*

Proof. If D is a PVMD such that every t-locally principal ideal of D is t-invertible then by Proposition 4 every integral t-invertible t-ideal of D is contained in at most a finite number of mutually t-comaximal t-invertible t-ideals of D. But by Proposition 1, D is of finite t-character. Conversely if D is of finite t-character then every t-locally principal ideal is t-invertible follows from Lemma 2.2 of [1]. □

In the following we present some Bazzoni-like statements for domains that are not PVMD's.

Proposition 6. *Let D be a domain such that every maximal t-ideal M of D contains a t-invertible t-ideal A such that A is contained in no other maximal t-ideal and for every $x \in M$ there is a t-invertible t-ideal containing $A + xD$. Then D is of finite t-character if and only if every t-locally principal t-ideal is t-invertible.*

Proof. Suppose that every t-locally principal t-ideal of D is t-invertible and suppose that x is a nonzero element of D that belongs to an infinite set $\{M_i\}_{i \in N}$, $M_i \neq M_j$ if $i \neq j$, of maximal t-ideals of D. For each i let A_i be a t-invertible t-ideal that belongs to only M_i.Then by the condition there is a t-invertible t-ideal H_i containing both A_i and x. Clearly H_i belongs only to M_i. In other words, for each i there is a t-invertible t-ideal $H_i \subseteq M_i$ such that $H_i \supseteq x$. Clearly, for $i \neq j$ $(H_i + H_j)_t = D$, because H_i and H_j do not share a maximal t-ideal. Consider the sequence of ideals $H_1^{-1}x$, $(H_1^{-1}H_2^{-1})_t x$, ... $(\prod_{i=1}^{i=n} H_i^{-1})_t x$ ... . Using the same argument as in

Proposition 4 we can show that the ideal $F = \sum_{n=1}^{\infty}(\prod_{i=1}^{i=n} H_i^{-1})_t x = \bigcup_{n=1}^{\infty}(\prod_{i=1}^{i=n} H_i^{-1})_t x$ is t-locally principal yet not invertible, a contradiction. Conversely if D is of finite t-character then every t-locally principal ideal is t-invertible follows from Lemma 2.2 of [1]. □

Corollary 2. *Let D be such that every maximal t-ideal of D is t-invertible (invertible, principal). Then D is of finite t-characher if and only if every t-locally principal ideal is t-invertible.*

To establish that there do exist non PVMD domains that meet the requirements of Proposition 6 and Corollary 2 we state the following result.

Proposition 7. *Let D be a PID, let L be a proper algebraic extension of the quotient field of D and let X be an indeterminate over L. Then the ring $R = D + XL[X]$ is of finite (t-)character if and only if D is a semilocal PID. In this case R is not a PVMD.*

Proof. From [13, page 107] we can conclude that every maximal ideal of R is of the form pR, where p is either a prime of D or a prime of the form $(1 + Xf(X))$. But it can be easily checked that every nonzero principal prime ideal is a maximal t-ideal and in this case a maximal ideal. Now if D is not a semilocal PID, then X belongs to all the prime ideals of the form $P + XDL[X]$, where P is a nonzero (principal) prime ideal of. So if there are infinitely many mutually comaximal principal primes say $\{p_1R, p_2R, ...\}$ then because $X \in p_iR$ we can set up an ideal F, as in Theorem 1 that is locally principal yet not invertible in clear contradiction of R being of finite (t-) character. Conversely let D be a semilocal PID. A typical nonzero element $g \in R$ is of the form $lX^r(1 + Xh(X))$, where $l \in L$. If $r = 0$ we get $g = a(1 + Xh(X))$, where $a \in D$. Again from [13] it can be established that, if $r > 0$, lX^r and $(1 + Xh(X))$ are comaximal, $(1 + Xh(X))$ is a product of primes (that generate maximal ideals of height 1) and the maximal (t-) ideals lX^r belongs to are only of the form $P + XL[X] = pR$ which are finite in number. Finally if $r = 0$, $g = a(1+Xf(X))$ and g is a product of primes that generate maximal ideals. That R is not a PVMD follows from the fact that R is not integrally closed. □

Now a word about r-Prüfer monoids. In [9] Houston, Malik and Mott introduced the notion of a "$*$-multiplication domain", for a finite character star operation $*$, as a domain whose nonzero finitely generated ideals are all $*$-invertible. But for a finite character star operation a $*$-invertible $*$-ideal is a t-invertible t-ideal [14, Theorem 1.1]. So, in a $*$-multiplication domain of [9] for every nonzero finitely generated ideal A we have $A^* = A_v$ and as we concluded in the Prüfer domain case we have $A^* = A_t$ for all $A \in F(D)$. To sum up, a $*$-multiplication domain of [9] is a PVMD. These domains have been extensively studied in literature as P$*$MD's (Prüfer $*$-multiplication domains), even for semistar operations see e.g. [4] and references there. In [7, Ch. 17], Halter-Koch translated the $*$-multiplication monoids as r-Prüfer monoids in the language of semigroups and ideal systems. So, for the ideal system r of finite character, an r-Prüfer monoid is a t-Prüfer monoid which is a monoid counterpart of a PtMD which is just the PVMD. In short, for $*$ (respectively, r) of finite character a P$*$MD (resp., r-Prüfer monoid) is a specialization of a PVMD (resp., t-Prüfer monoid). So any result proved for PVMD's (resp t-Prüfer monoids) can be verified for P$*$MD's in the same manner as we did for Prüfer domains. Then these results can be translated, in the usual manner to r-Prüfer monoids with a wider area of application.

Finally, let us note that there are Noetherian domains with some nonzero element x in an infinite number of maximal ideals. Looking at the above results it appears that maximal t-ideals have more control. So, here's a question: Is there a domain D that is not of finite t-character yet has the property that every nonzero t-locally principal ideal is t-invertible?

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