COMMUNICATIONS IN ALGEBRA, 17(11), 2835-2852 (1989)

t-LINKED OVERRINGS AND PRÜFER v-MULTIPLICATION DOMAINS

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1. Introduction.

Let R be a (commutative integral) domain. Among the numerous overring-theoretic characterizations of Prüfer domains is the following result of Davis [D, Theorem 1]: R is a Prüfer domain if and only if each overring of R is integrally closed. One goal of our work, realized in Theorem 2.10 below, is to obtain an analogous characterization of Prüfer v-multiplication domains, or PVMD's for short. (A PVMD is a domain R such that R_P is a valuation domain for each prime t-ideal P of R. A convenient reference for PVMD's is [MZ1].) To this end, we make the following definition. An overring T of R is t-linked (over R) if, for each finitely generated fractional ideal A of R such that $A^{-1} = R$, one has $(AT)^{-1} = T$. (Proposition 2.2 identifies several families of t-linked overrings; the connection with the t-operation is indicated in Proposition 2.1.) As a generalization of [H, Proposition 1.6], it was shown by Kang [K1, Theorem 3.8 and Corollary 3.9] that each t-linked overring of a PVMD is itself a PVMD (and, hence, integrally closed). Theorem 2.10 gives the following generalization of Kang's result: R is a PVMD if (and only if) each t-linked overring of R is integrally closed.

Additional connections between "t-linked" and "PVMD" are given. For instance, Corollary 2.4 establishes that the complete integral closure of a PVMD is (t-linked and hence) a PVMD. As for whether the integral closure of R is t-linked, we give an affirmative answer in the Noetherian (indeed, quasicoherent) case: see Corollary 2.14 (a). In addition, Corollary 2.18 and Remark 2.19 serve to identify the proper subclass of PVMD's R characterized by "each t-linked overring of R is R-flat." This assertion is to be contrasted with the result of F. Richman [R, Theorem 4] that "each overring of R is R-flat" characterizes the Prüfer domains R. Also noteworthy (and useful) is Theorem 2.6 characterizing the domains R such that every overring of R is t-linked.

Despite the above emphasis on Prüferian motivation, it should be noted that Krull domains form another important class of PVMD's. Their relevance to a "t-linked" study is clear since there exist numerous characterizations of Krull domains in terms of the t-operation (cf. [MMZ], [MZ2], [HZ], [K2], and [Ga]). This motivation is felt in Remark 2.8 (b) and in two new characterizations of Krull domains (see Corollary 2.22).

Throughout, R denotes a domain with integral closure R', complete integral closure R^*, and quotient field K. The set of all (resp., all finitely generated) nonzero fractional ideals of R is denoted by F(R) (resp., by f(R)).

2. Results.

Before characterizing t-linked overrings, we shall provide a brief review of two important star-operations on the fractional ideals of a domain R. For $A \in F(R)$, we define $A_v = (A^{-1})^{-1}$ and $A_t = \cup F_v$, where F ranges over the finitely generated fractional ideals contained in A. It is easy to see that $A_t = A_v$ if $A \in f(R)$. We say that A is divisorial or a v-ideal if $A_v = A$; and that A is a t- ideal if $A_t = A$. Since $A \subset A_t \subset A_v$ in general, each divisorial ideal is a t-ideal. By a t-prime (of R), we mean a prime ideal of R which is also a t-ideal. It is known that each height 1 prime ideal is a t-prime. More generally, any ideal of the form $(Rb:_R a)$, with a and b in $R \setminus \{0\}$, is a t-ideal; any associated prime, that is any prime minimal over such an ideal $(Rb:_R a)$, is a t-prime; and, in fact, any prime minimal over a t-ideal is a t-prime. Moreover, any t-ideal is contained in a maximal t-ideal, which is necessarily prime. Proofs of these and other facts about star-operations may be found in [G, Sections 32 and 34] and in [J].

PROPOSITION 2.1. Let T be an overring of a domain R. Then the following conditions are equivalent:

(1) T is t-linked over R; that is, if $A \in f(R)$ and $A^{-1} = R$, then $(AT)^{-1} = T$;

(2) If $A \in f(R)$ and $A_t = R$, then $(AT)_t = T$;

(3) If Q is a t-prime of T, then $(Q \cap R)_t \neq R$.

Proof. (1) $\Leftrightarrow$ (2): It suffices to note that if $B \in f(R)$, then $B^{-1} = R \Leftrightarrow B_t = R$. (Indeed, $B^{-1} = R \Rightarrow B_t = B_v = (B^{-1})^{-1} = R^{-1} = R$; and $B_t = R \Rightarrow B^{-1} = (B_v)^{-1} = (B_t)^{-1} = R^{-1} = R$.)

(2) $\Rightarrow$ (3): Deny. Then there is a t-prime Q of T such that $(Q \cap T)_t = R$; so, $A_v = A_t = R$ for some $A \in f(R)$ such that $A \subset Q \cap R$. By (2), $(AT)_t = T$. However, $(AT)_t \subset Q_t = Q \subsetneqq T$, a contradiction.

(3) $\Rightarrow$ (2): Deny. Then there exists $A \in f(R)$ such that $A_t = R$ and $(AT)_t \neq T$. By the above remarks, the t-ideal $(AT)_t$ lies in some maximal t-ideal Q of T. Since $A \subset (AT)_t \cap R \subset Q \cap R \subset R$ and $A_t = R$, it follows that $(Q \cap R)_t = R$, contradicting (3). $\square$

We next give some examples of t-linked overrings.

PROPOSITION 2.2. Let R be a domain. Then:

(a) Any directed union of t-linked overrings of R is t-linked over R.

(b) Any intersection of t-linked overrings of R is t-linked over R.

(c) Any R-flat overring of R is t-linked over R.

(d) Any generalized transform of R is t-linked over R.

(e) If $A \in F(R)$, then $(A_v :_K A_v)$ is t-linked over R.

Proof. (a) Let $\{T_\alpha\}$ be a directed set of t-linked overrings of R, and consider the overring $T = \cup T_\alpha$. Suppose $A \in f(R)$ satisfies $A^{-1} = R$. We shall show that $(AT)^{-1} = T$. Note that $(AT_\alpha)^{-1} = T_\alpha$ for each α, since T_α is t-linked. Write $A = Ra_1 + Ra_2 + \cdots + Ra_k$, and consider any $u \in (AT)^{-1}$. For each i, $1 \leq i \leq k$, there exist α_i such that $ua_i \in T_{\alpha_i}$. By directness, there exists α such that $ua_i \in T_\alpha$ for all i, and so $uA \subset T_\alpha$. Hence $u \in (AT_\alpha)^{-1} = T_\alpha \subset T$. This proves that $(AT)^{-1} \subset T$. The reverse inclusion is evident since $A^{-1} = R$ implies that $A \subset A_v = R^{-1} = R$.

(b) Consider $T = \cap T_\alpha$, where each T_α is a t-linked overring of R. Suppose $A \in f(R)$ satisfies $A^{-1} = R$. We shall show that if $u \in (AT)^{-1}$, then $u \in T$. Since $uA \subset T \subset T_\alpha$ for each α, we have $u \in (AT_\alpha)^{-1} = T_\alpha$; thus, $u \in \cap T_\alpha = T$.

(c) Let T be an R-flat overring of R. Suppose $A \in f(R)$ satisfies $A^{-1} = R$. Write $A = Ra_1 + Ra_2 + \cdots + Ra_k$, with each $a_i \neq 0$. Then $(AT)^{-1} = \{u \in K : ua_i \in T \text{ for each } i\} = \cap Ta_i^{-1}$; so by flatness (cf. [Je, Theorem 1]), this intersection is just $T(\cap Ra_i^{-1})$. Hence, $(AT)^{-1} = TA^{-1} = TR = T$.

(d) This is due to Kang [K1, Lemma 3.10], and is included here for the sake of completeness.

(e) Put $T = (A_v :_K A_v)$, and suppose $B \in f(R)$ satisfies $B^{-1} = R$. We shall show that if $u \in (BT)^{-1}$, then $u \in T$. Indeed, $uB \subset T$ and so $uBA_v \subset A_v$. By applying the v-operation, we find $uA_v \subset A_v$, whence $u \in (A_v :_K A_v) = T$. □

COROLLARY 2.3. If R is a domain, then R^* (the complete integral closure of R) is t-linked over R.

Proof. If A and B are in $F(R)$, it is straightforward to check that $(A_v :_K A_v)$ and $(B_v :_K B_v)$ are each contained in $((AB)_v :_K (AB)_v)$. By Proposition 2.2 (e), it follows that R^* is a directed union of t-linked overrings of R. (See [F, Lemma 3.1] and observe that $(A :_K A) \subset (A_v :_K A_v)$ for $A \in F(R)$.) The assertion now follows from Proposition 2.2 (a). □

COROLLARY 2.4. The complete integral closure of a PVMD is a PVMD.

Proof. Combine Corollary 2.3 with the result of Kang [K1, Theorem 3.8 and Corollary 3.9] mentioned in the introduction. □

REMARK 2.5. In the definition of "t-linked," there is no *a priori* reason to take the extension domain T to be an overring of R. We have chosen the simpler setting of overrings for this paper because of the intended applications. Some of what we do here, such as the (role of [Je, Theorem 1] in the) proof of Proposition 2.2 (c), carries over to the expanded context.

For any domain R, both R and K are t-linked over R. Thus, if R is a valuation domain of (Krull) dimension at most 1, all its overrings are t-linked (since R and K are its *only* overrings). Which domains R have the property that each overring of R is t-linked? The next two results answer this question, identifying a class containing all valuation domains, all one-dimensional domains, and more.

THEOREM 2.6. For a domain R, the following conditions are equivalent:

(1) Each overring of R is t-linked over R;

(2) Each valuation overring of R is t-linked over R;

(3) Each maximal ideal of R is a t-ideal;

(4) Each proper nonzero ideal I of R satisfies $I_t \neq R$;

(5) Each proper nonzero finitely generated ideal I of R satisfies $I_t \neq R$;

(6) Each t-invertible ideal of R is invertible.

Proof. Each of the implications (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (5) is trivial.

(2) $\Rightarrow$ (3): Deny. Choose a maximal ideal M of R such that M is not a t-ideal. As $M \subsetneqq M_t \subset R$, it follows that $M_t = R$, and so $A_t = R$ for some finitely generated ideal $A \subset M$. Next, choose a valuation overring (V,N) of R such that $N \cap R = M$ (cf. [Kp, Theorem 56]). Since finitely generated nonzero ideals of a valuation domain are principal (and hence divisorial), $(AV)_t \subset N_t = N \subsetneqq V$. In view of Proposition 2.1, this contradicts the assumption that V is t-linked over R.

(3) $\Rightarrow$ (1): Deny. By Proposition 2.1, there exists an overring T of R and a t-prime Q of T such that $(Q \cap R)_t = R$. On the other hand, $Q \cap R$ lies in some maximal ideal M of R, and $M_t = M$ by (3). Hence, $R = (Q \cap R)_t \subset M_t = M \subsetneqq R$, a contradiction.

(5) $\Rightarrow$ (6): Let I be a t-invertible ideal of R; that is, $(II^{-1})_t = R$. So $A_t = R$ for some finitely generated $A \subset II^{-1}$. By (5), $A = R$, and so $II^{-1} = R$; that is, I is invertible.

(6) $\Rightarrow$ (4): Deny. Hence some proper nonzero ideal I of R satisfies $I_t = R$. Then I is t-invertible, hence invertible by (6), hence divisorial, and hence a t-ideal. In other words, $I_t = I$, contradicting $I_t = R$. □

We next recall some background from [Do]. Let R be a domain. Then R is said to be a going-down domain (resp., treed) if $R \subset T$ satisfies going-down for each overring T of R (resp., if Spec (R), as a poset under inclusion, is a tree). Each Prüfer domain is a going-down domain; so is each domain of dimension at most 1. Each going-down domain is treed; the converse is false.

COROLLARY 2.7. Let R be a domain. Then each overring of R is t-linked over R in each of the following cases:

(a) R is a Prüfer domain;

(b) dim (R) ≤ 1;

(c) R is a going-down domain;

(d) R is treed.

Proof. By the above remarks, (a) $\Rightarrow$ (c), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (d). So, we may assume that R is treed. By Theorem 2.6, it will suffice to show that each maximal ideal M of R is a t-ideal. For each $m \in M \setminus \{0\}$, there exists an associated (t-) prime P such that $m \in P \subset M$. Since R is treed, M is therefore a directed union of t-primes P_α. If $M_t = R$, then $1 \in (\cup P_\alpha)_t$ and so, by directedness, $1 \in (P_\alpha)_t$ for some α, contradicting $(P_\alpha)_t = P_\alpha \subsetneq R$. Hence $M_t \neq R$; that is, $M_t = M$. □

REMARK 2.8. (a) Suppose that R is a domain such that whenever $A \in f(R)$ satisfies $A^{-1} = R$ then $(a,b)_t = R$ for some $a, b \in A$. In this case, the conditions in Theorem 2.6 are also equivalent to "each simple overring of R is t-linked." To see this, suppose that a proper two-generated $I = (a,b)$ of R satisfies $I_t = R$; our task is to find a simple overring T of R which is not t-linked. To this end, find a valuation overring V of R in which I survives (cf. [Kp, Theorem 56]). Without loss of generality, $IV = bV$ since V is valuation. We shall show that the simple overring $T = R[a/b]$ is not t-linked. Indeed, $(IT)^{-1} = (bT)^{-1} = b^{-1}T \neq T$, although $I^{-1} = (I_v)^{-1} = (I_t)^{-1} = R^{-1} = R$.

(b) It is known (cf. [M, Theorem 1]) that a domain R is a Prüfer domain if and only if R is an integrally closed treed domain such that $Ra \cap Rb$ is finitely generated for all a, b $\in$ R. Despite Corollary 2.7, there exists an integrally closed domain R such that each maximal ideal of R is a t-ideal, for each a and b in R there exists $A \in f(R)$ such that $Ra \cap Rb = A_t$, and R is not a Prüfer domain. For such an example consider a two-dimensional integrally closed Mori domain each of whose maximal ideals is divisorial. (A Mori domain is one which satisfies the ascending chain condition on divisorial ideals.) An explicit example is given by $(K + XK[X,Y])_{XK[X,Y]}$ (cf. [BG, Example 4.6 (b)]).

(c) It is well known (cf. [MZ1, Proposition 4.4]) that a domain R is a Prüfer domain if (and only if) R is a treed PVMD. In fact, R is a Prüfer domain if (and only if) R is a treed P-domain. (Recall that R is said to be a <u>P-domain</u> if R_Q is a valuation domain for each associated prime Q of R.) To see this, we show that R_P is a valuation domain for each nonzero P $\in$ Spec (R). Note that $R_P = \cap (R_P)_{QR_P} = \cap R_Q$, where Q ranges over the associated primes contained in P (cf. [Kp, Exercise 20, page 42]). By the hypotheses, R_P is the intersection of a chain of valuation domains, and hence is valuation. □

We next record a key step in our path to generalizing Kang's result.

<u>PROPOSITION 2.9.</u> Let R be a domain, P a t-prime of R, and T an overring of R. Then $T_{R\setminus P}$ is t-linked over R.

<u>Proof.</u> We shall verify condition (3) in Proposition 2.1. Let Q be a (not necessarily t-) prime of $T_{R\setminus P}$. Since $Q \cap R \subset P$, we have $(Q \cap R)_t \subset P_t = P \subsetneq R$. □

The next result is our generalization of [K1, Corollary 3.9] and analogue of [D, Theorem 1]. First, recall (cf. [GHt]) that a domain R is called <u>seminormal</u> if, whenever $u \in K$ satisfies $u^2 \in R$ and $u^3 \in R$, then $u \in R$.

THEOREM 2.10. For a domain R, the following conditions are equivalent:

(1) Each t-linked overring of R is integrally closed;

(2) R is integrally closed and each t-linked overring of R is seminormal;

(3) R is a PVMD.

Proof. (3) $\Rightarrow$ (1): As explained in the introduction, this follows from [K1, Theorem 3.8]. However, we offer the following alternate approach, making use of techniques that will see service again in Proposition 2.13. It suffices to show that T_Q is a valuation domain for each t-linked overring T of R and each t-prime Q of T. Using Proposition 2.1, we find a (maximal) t-prime P of R such that $(Q \cap R)_t \subset P$. Then T_Q is an overring of the valuation domain R_P, and so is itself a valuation domain.

(1) $\Rightarrow$ (2): R is t-linked over R; and integrally closed $\Rightarrow$ seminormal.

(2) $\Rightarrow$ (3): We shall show that R is a PVMD by verifying that R_P is a valuation domain for each t-prime P of R. It suffices to show that if $u \in K \setminus \{0\}$, then either u or u^{-1} is in R_P. By Proposition 2.9, $A = R[u^2,u^3]_{R\setminus P}$ is t-linked over R, and hence seminormal by (2). It follows that $u \in A$, since both u^2 and u^3 are in A. Viewing A as $R_P[u^2,u^3]$, we see that $h(u) = 0$, for some $h \in R_P[X]$ such that the coefficient of X in h is 1. Since R_P is integrally closed, the (u,u^{-1})-Lemma (cf. [Kp, Theorem 67]) yields that either u or u^{-1} is in R_P. □

The literature contains several examples of a P-domain R (indeed a locally PVMD) which is not a PVMD. One such is given in [MZ1, Example 2.1]. It can be shown that every maximal ideal of this R is a t-ideal. Thus, by Theorem 2.6, each overring of this R is t-linked; but, by Theorem 2.10, not every (t-linked) overring of R is seminormal.

Recall from [GH2] that if R is a domain, then <u>the unique minimal overring S of</u> R is (if it exists) a proper overring S of R such that $S \subset T$ for each proper overring T of R.

COROLLARY 2.11. For a domain R, the following conditions are equivalent:

(1) Each proper t-linked overring of R is integrally closed;

(2) Either (a) R is a PVMD or
(b) R has a unique minimal overring S and S is a Prüfer domain. (Moreover, in this case, R is quasilocal.)

Proof. (2) $\Rightarrow$ (1): Given (a), apply [K1, Theorem 3.8] or Theorem 2.10. Given (b), notice that each proper overring of R is an overring of S, hence is a Prüfer domain (cf. [G, Theroem 26.1]), and hence is integrally closed.

(1) $\Rightarrow$ (2): By Theorem 2.10, we may assume that R is not integrally closed. Now, write $R = \cap R_P$, where P ranges over the associated primes of R. If each such R_P is distinct from R, it follows from Proposition 2.2 (c) and (1) that R is the intersection of integrally closed overrings, contradicting $R \neq R'$. So, (R,M) is quasilocal for some associated (hence, t-) prime M of R. By the implication (3) $\Rightarrow$ (1) in Theorem 2.6, each overring of R is t-linked. Thus, by (1), each proper overring of R is integrally closed. Such R have been catalogued (see either [MG, Theorems 10, 12, and 13] or [AADH, Remark 3.3 (b)]). Applying these classification results yields (b). □

Condition (2) in Theorem 2.10 is reminiscent of [ADH, Theorem 2.3], characterizing the domains each of whose overrings is seminormal. We next record another "t-linked" variant of such work.

COROLLARY 2.12. Let R be a domain such that each t-linked overring of R is seminormal. Put $T = \cap R'_{R\setminus P}$, where P ranges over the t-primes of R. Then T is a PVMD.

Proof. We shall show that T_Q is a valuation domain for each t-prime Q of T. In fact, we shall show that if $u \in K\backslash\{0\}$, then either u or u^{-1} is in T_Q. First, note via Proposition 2.9 and Proposition 2.2 (b) that T is t-linked over R. Moreover, by Proposition 2.9, $A = T_Q[u^2,u^3] = T[u^2,u^3]_{T\backslash Q}$ is t-linked over (T and hence over) R. So, by hypothesis, A is seminormal. Therefore $u \in A$ and, as in the proof of Theorem 2.10, T_Q contains either u or u^{-1}. □

Next, we further investigate the ring T constructed in Corollary 2.12, in order to shed light on whether R′ is t-linked over R.

PROPOSITION 2.13. Let R be a domain and T an overring of R. Then:

(a) T is t-linked over R if and only if $\cap T_{R\backslash P} = T$, where P ranges over the t-primes of R.

(b) R has a smallest integrally closed t-linked overring, namely $\cap R'_{R\backslash P}$, where P ranges over the t-primes of R.

Proof. (a) The "if" assertion is direct via Propositions 2.9 and 2.2 (b). Conversely, suppose that T is t-linked over R, and put $A = \cap T_{R\backslash P}$, where P ranges over the t-primes of R. If Q is a t-prime of T, it follows from condition (3) in Proposition 2.1 that $(Q \cap R)_t \neq R$, and so $Q \cap R \subset P$ for some t-prime P of R. Hence $R\backslash P \subset T\backslash Q$, yielding $T_{R\backslash P} \subset T_Q$. Since $T = \cap T_Q$, we have $A \subset T$. As the reverse inclusion is evident, $A = T$.

(b) Let B be integrally closed and t-linked over R. Then $B \supset R'$, whence by (a), $B = \cap B_{R\backslash P} \supset \cap R'_{R\backslash P}$. Of course, $\cap R'_{R\backslash P}$ is integrally closed and, as in the proof of Corollary 2.12, also t-linked over R. □

Recall from [BAD] that an integral domain R is said to be quasicoherent if each intersection of finitely many principal ideals of R is finitely generated. Each coherent domain is quasicoherent; hence, so is each Noetherian domain.

COROLLARY 2.14. Let R be a domain. Then:

(a) If R is quasicoherent, then R' is t-linked over R.

(b) If R is quasicoherent and each t-linked overring of R is seminormal, then R' is a PVMD.

Proof. (a) We modify an argument of Beck (cf. [F, Lemma 4.5]) originally designed for the Noetherian case. We shall show that if $A = Ra_1 + \cdots + Ra_k \in f(R)$ satisfies $A^{-1} = R$, then $(AR')^{-1} = R'$. Consider $u \in (AR')^{-1}$. As $uA \subset R'$, each a_i leads to a finitely generated ideal I_i of R such that $ua_iI_i \subset I_i$ (cf. [Kp, Theorem 12]). Note that $I = \prod I_i$ is finitely generated. Moreover, $ua_iI \subset I$ for each i; therefore $uAI \subset I$. Applying the v-operation, we find that $uI_v \subset I_v$. In addition, quasicoherence assures that I_v is finitely generated (cf. [BAD, page 1122]). Another application of [Kp, Theorem 12] yields that $(I_v :_K I_v) \subset R'$, and so $u \in R'$, as desired.

(b) By (a) and Proposition 2.13 (a), $\cap R'_{R\setminus P} = R'$, where P ranges over the t-primes of R. The assertion now follows from Corollary 2.12. □

REMARK 2.15. For any domain R, the pseudo-integral closure of R is defined to be $\tilde{R} = \bigcup_{A \in f(R)} (A_v :_K A_v)$. It is easy to see that $R' \subset \tilde{R} \subset R^*$; it can also be shown that $\tilde{R}$ is integrally closed. Just as in the proof of Corollary 2.3, one shows that $\tilde{R}$ is t-linked over R. Hence, via Theorem 2.10, we see that if each t-linked overring of R is seminormal, then $\tilde{R}$ is a PVMD. Pseudo-integral closure is the subject of a manuscript in preparation by D. F. Anderson and the second- and fourth-named authors.

Our next goal is a "t-linked" analogue of Richman's flat-theoretic characterization of Prüfer domains [R, Theorem 4]. This will be given in Corollary 2.18. First, we give two results about overrings of a PVMD. It will be convenient to let a subintersection of a PVMD, R, mean any intersection $\cap R_P$, where P ranges over a set of some prime t-ideals of R. Kang [K1, Theorem 3.8] proved that an overring T of a PVMD, R, is a subintersection of R $\Leftrightarrow$ T is t-linked over R.

PROPOSITION 2.16. Let R be a PVMD, let $A \in f(R)$, and let T be a subintersection of R. Then $(A_vT)_{v_T} = (AT)_{v_T}$, where v_T denotes the v-operation on the fractional ideals of T.

Proof. $R = \cap R_{P_\beta}$, where $\{P_\beta\}$ is the set of all t-primes of R. By hypothesis, $T = \cap R_{Q_\alpha}$, where $\{Q_\alpha\} \subset \{P_\beta\}$; this representation induces a star-operation which we denote by $*$. (See [G, Theorem 32.5].) Now

$$(A_vT)_* = \cap A_vTR_{Q_\alpha} = \cap ATR_{Q_\alpha} = (AT)_*,$$

where the second equality follows easily from the fact that R_{Q_α} is a valuation domain. Applying v_T to the displayed equation yields $(A_vT)_{v_T} = (AT)_{v_T}$, as asserted. □

PROPOSITION 2.17. Let R be a PVMD and T a subintersection of R. Then T is R-flat if and only if A_vT is divisorial in T for each $A \in f(R)$.

Proof. Assume T is R-flat. It is known (see the proof of Proposition 2.2 (c)) that $(BT)^{-1} = B^{-1}T$ for all finitely generated fractional ideals B of R. Now, let $A \in f(R)$. Since R is a PVMD, $A^{-1} = B_v$ for some finitely generated B. Then $A_vT = (B_v)^{-1}T = B^{-1}T = (BT)^{-1}$, which is divisorial.

Conversely, suppose that A_vT is divisorial for each $A \in f(R)$. We shall show that T is R-flat; it suffices (cf. [F, Lemma 6.5]) to show that $(AT)^{-1} = A^{-1}T$ for all $A \in f(R)$. As above, $A^{-1} = B_v$ for some finitely generated B; and we let $*$ denote the star-operation induced by the representation $T = \cap R_{P_\alpha}$. Then $(AT)^{-1} = ((AT)^{-1})_* = \cap (AT)^{-1}R_{P_\alpha} = \cap (AR_{P_\alpha})^{-1} = \cap A^{-1}TR_{P_\alpha} = (A^{-1}T)_* = (B_vT)_*$. However, B_vT is divisorial by virtue of the hypothesis and, a fortiori, coincides with $(B_vT)_*$. Therefore, $A^{-1}T = B_vT = (B_vT)_* = (AT)^{-1}$. □

By reasoning as above, one may prove the following result. Let T be an overring of the PVMD R. Then T is R-flat if and only if $(AT)_{v_T} = A_vT$ for each $A \in f(R)$. This result does not seem to be known even in the well-studied case of Krull domains.

COROLLARY 2.18. Let R be a domain. Then each t-linked overring of R is R-flat if and only if R is a PVMD such that $A_vT = (AT)_{v_T}$ for each $A \in f(R)$ and each subintersection T of R.

Proof. The "if" half follows directly from the assertions in Proposition 2.17. Conversely, assume that each t-linked overring of R is R-flat. Let P be an associated prime of R. Then the maximal ideal of R_P is an associated prime, hence a t-ideal. By the implication (3) $\Rightarrow$ (1) in Theorem 2.6, each overring T of R_P is t-linked over R_P, and hence t-linked over R (using Proposition 2.2 (c)). Thus T is R-flat and, by [R, Lemma 2], therefore R_P-flat. By [R, Theorem 4], R_P is a Prüfer (and, hence, valuation) domain. Thus R is a P-domain. By Propositions 2.16 and 2.17, it remains only to prove that R is a PVMD. By Theorem 2.10, it suffices to show that each t-linked overring S of R is integrally closed. Since S is R-flat, S is an intersection of localizations of R by [R, Corollary, page 795], and so Proposition 2.2 (b), (c) yields that S is integrally closed. □

REMARK 2.19. The equivalent conditions in Corollary 2.18 do not characterize arbitrary PVMD's. One need only consider an example, such as [F, page 32], involving a Krull domain (hence PVMD) R having a subintersection (hence t-linked overring) T such that T is not R-flat.

We shall close with some applications to Krull domains. For partial motivation, recall that [BD, Section 3] studied the domains each of whose overrings is a Mori domain.

PROPOSITION 2.20. Let R be a domain such that each t-linked overring of R is a Mori domain. Then:

(a) Each divisorial prime of R has height 1.

(b) R^*, the complete integral closure of R, is a Krull domain.

<u>Proof</u>. (a) Let M be a divisorial prime of R. By [HLV, Proposition 1.1 (iv)], MR_M is a divisorial (hence, t-) prime of R_M. As R_M inherits the hypotheses (cf. Proposition 2.2 (c)), it follows via Theorem 2.6 that each overring of R_M is a Mori domain. Thus, each non-trivial valuation overring of R_M is a (one-dimensional) DVR [BD, Theorem 3.4], whence [G, Proposition 43.16] yields that $1 = \dim(R_M) = \mathrm{ht}(M)$.

(b) By Corollary 2.3 and the hypothesis, R^* is Mori, and so it suffices to show that R^* is completely integrally closed. By (a), $R^* = \cap R^*_M$, where M ranges over height 1 primes of R^*. As R^* is integrally closed [G, Theorem 13.1], so is each R^*_M; hence R^*_M is an intersection of valuation overrings V_α. As in the proof of (a), each V_α is a DVR, hence completely integrally closed. Thus R^* is an intersection of completely integrally closed overrings. □

<u>REMARK 2.21</u>. Neither of the implications in Proposition 2.20 can be reversed. Indeed, consider the one-dimensional domain $D = \mathbb{Q} + X\mathbb{R}[X]_{(X)}$. The complete integral closure of D is $\mathbb{R}[X]_{(X)}$, which is a Krull domain. By Corollary 2.7 (b), each overring of D is t-linked over D. In addition, it is easy to show that D' is not a Dedekind domain. Therefore, it follows from [BD, Theorem 3.4] that some (t-linked) overring T of D is not a Mori domain. In fact, $\mathbb{Q}[\pi] + X\mathbb{R}[X]_{(X)}$ is such a T.

<u>COROLLARY 2.22</u>. For a domain R, the following conditions are equivalent:

(1) R is integrally closed and each t-linked overring of R is a Mori domain;

(2) R is a Mori domain and each t-linked overring of R is integrally closed;

(3) R is a Krull domain.

Proof. (3) ⇒ ((1) and (2)): Let R be a Krull domain and T a t-linked overring of R. By [K1, Theorem 3.8], T is a subintersection of R. By [F, Corollary 1.5], T is a Krull domain; hence, T is a (completely) integrally closed Mori domain.

(2) ⇒ (3): Couple Theorem 2.10 with the fact that any Mori PVMD is a Krull domain (cf. [Z, Corollary 2.2]).

(1) ⇒ (3): By (1), R is a Mori domain. Moreover, R is completely integrally closed since, by adapting the proof of Proposition 2.20 (b), we see that R is an intersection of rank one DVR overrings. Hence, R is a Krull domain. □

In closing, we point out the following consequence of Corollary 2.22. If R is a domain such that R′ is t-linked over R and each t-linked overring of R is a Mori domain, then R′ is a Krull domain. Along with (2.12) - (2.15), this reinforces the importance of the open question as to whether R′ is t-linked over R for each domain R. Notice that this is altogether different from the question whether, for each (not necessarily finitely generated) ideal A of R, $A^{-1} = R$ implies that $(AR')^{-1} = R'$. The latter question is answered in the negative by considering the example $R = k[\{X^{2n+1}Y^{n(2n+1)}\}_{n=0}^{\infty}]$ (cf. [GH1]), which has integral closure $R' = k[\{XY^{n}\}_{n=0}^{\infty}]$. It is not difficult to show in this example that the maximal ideal $M = (\{X^{2n+1}Y^{n(2n+1)}\}_{n=0}^{\infty})$ satisfies $M^{-1} = R$ and $(MR')^{-1} \neq R'$.

ACKNOWLEDGMENTS

The first author gratefully acknowledges the hospitality and partial support offered by the University of North Carolina at Charlotte during his enjoyable two-week visit in May, 1988. He was also supported in part by a University of Tennessee Faculty Research Grant.

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Received: June 1988