1

What v-coprimality can do for you

Muhammad Zafrullah

No Institute Given

Address: 57-Colgate Street, Pocatello, ID 83201

1.1 Introduction

Let D be an integral domain with quotient field K. Two elements $x, y \in D \backslash \{0\}$ are said to be v-coprime if $xD \cap yD = xyD$. A saturated multiplicative set $S \subseteq D \backslash \{0\}$ is a splitting set of D if every $x \in D \backslash \{0\}$ can be written as $x = ds$ where $s \in S$ and d is v-coprime to every member of S. The notions of v-coprimality and splitting sets can be traced back to the work of Gilmer and Parker [31] and Mott and Schexnayder [34]. These authors worked on generalizing the following theorem due to Nagata [35]: Let D be a Noetherian domain and let S be a multiplicative set generated by principal nonzero primes of D. If D_S is a UFD then so is D . The purpose of this article is to present a brief survey of the notion of v-coprimality, its applications, its morphs and its generalizations; as a bouquet of flowers from the garden that sprang up from the seeds planted by Gilmer, Mott, Parker and Schexnayder. The space constraints make it hard to present the full view of the garden, but I will do my best to provide a sizeable bouquet. Before I get down to describing what I aim to do, it seems expedient to give a brief description of the tools that I will be using throughout this survey.

Let $F(D)$ be the set of nonzero fractional ideals of D, $A^{-1} = \{x \in K : xA \subseteq D\}$, $A_v = (A^{-1})^{-1} = \bigcap_{A \subseteq cD} cD$, $c \in K \backslash \{0\}$.

A function $*$ on $F(D)$ is called a star operation, if for all $a \in K \backslash \{0\}$ and $A, B \in F(D)$, the following hold.

(1*) $(a)^* = (a)$, $(aA)^* = aA^*$, (2*) $A \subseteq A^*$ and $A \subseteq B \Rightarrow A^* \subseteq B^*$ (3*) $(A^*)^* = A^*$.

Given that $*$ is a star operation on $F(D)$ and $A, B \in F(D)$, we have $(AB)^* = (A^*B)^* = (A^*B^*)^*$. These equations are said to define the "$*$-

multiplication". The function on $F(D)$ defined by $A \to A_v$ is a star operation such that for any star operation $*$ and for any $A \in F(D)$ we have $A^* \subseteq A_v$. To each star operation $*$ we can associate $*_f$ defined by $A^{*_f} = \cup\{F^* \colon 0 \neq F$ is a finitely generated D-submodule of $A\}$ for $A \in F(D)$. Call $*$ of finite type if $A^* = A^{*_f}$ for all $A \in F(D)$. Indeed for any star operation $*$ the operation $*_f$ is of finite type. The well known t-operation is given by $t = v_f$. (So, if A is finitely generated then obviously $A_t = A_v$.) The identity function $A \mapsto A$ on $F(D)$ is the d-operation. If $\{D_\alpha\}$ is a family of overrings of D such that $D = \cap D_\alpha$ then the function $A \mapsto A^* = \cap AD_\alpha$ is also a star operation. An integral ideal P of D is called a prime $*$-ideal if P is a $*$-ideal and a prime ideal. If $*$ is of finite type, a proper integral ideal M that is maximal with respect to being a $*$-ideal is called a maximal $*$-ideal and is necessarily prime. Moreover, every proper $*$-ideal is contained in at least one maximal $*$-ideal. For $*$ of finite type the set of maximal $*$-ideals is usually denoted by $*$-$Max(D)$. It can be shown that $D = \bigcap_{M \in *-Max(D)} D_M$. The star operation induced by $\{D_M\}_{M \in t\text{-}Max(D)}$ is usually denoted by w. Obviously every prime ideal contained in a maximal t-ideal is a w-ideal.

An ideal $A \in F(D)$ is a $*$-ideal of finite type if $A = B^*$ for some f.g. $B \in F(D)$, and A is $*$-invertible if $(AB)^* = D$ for some $B \in F(D)$. A t-invertible t-ideal is of finite type, and every invertible ideal is a v-ideal. D is a Prufer v-multiplication domain (PVMD) if each two generated nonzero ideal of D is t-invertible. Clearly a GCD domain is a PVMD. For details on star operations the reader may consult sections 32 and 34 of Gilmer's book [30] or, especially for the w-operation, the survey [44].

The paper is split into six sections. In section 2, I briefly treat the notion of v-coprimality. I indicate ways in which v-coprimality is different from coprimality. I also introduce the more general notion of $*$-coprimality by saying that two elements $x, y \in D$ are $*$-coprime if $(x, y)^* = D$ and characterize v-coprimality. Then I show that v-coprimality is similar to disjointness in partially ordered groups. In section 3, I show how v-coprimality has been used in circumstances, to do with divisibility, where coprimality has no effect. In section 4, I describe the splitting sets, indicating some properties. I also provide a brief historical background on them. In section 5, I give some examples and applications of splitting sets giving various forms and generalizations of Nagata-type theorems. In the 6th and last section, I indicate the kind of generalizations of splitting sets that have interested me and my co-workers.

1.2 v-coprimality

In this section I briefly treat the notion of v-coprimality. I show the ways in which v-coprimality is different from coprimality. I also introduce the more general notion of $*$-coprimality by saying that two elements $x, y \in D$ are

$*$-coprime if $(x, y)^* = D$ and characterize v-coprimality. Then I show that v-coprimality is similar to disjointness in partially ordered groups.

Definition 1. *Two nonzero elements $x, y \in D$ are called v-coprime if $(x, y)_v = D$ (i.e. $xD \cap yD = xyD$ or equivalently $(x, y)^{-1} = D$).*

It is easy to establish that $(a, b)_v = D \Leftrightarrow ((a, b) \subseteq (c/d) \Rightarrow c|d)$. So $(x, y)_v \neq D \Leftrightarrow$ (there exist $c, d \in D$ such that $c \nmid d$ but$(a, b) \subseteq c/d)$. Now, ordinarily $x, y \in D$ are said to be coprime if x and y have no nonunit common factor in D. Note that x, y being v-coprime implies x, y coprime but not conversely; as is apparent from the following discussion. Next, the negation of coprimality is much cleaner than the negation of v-coprimality. Indeed it is useful to note that $\text{GCD}(a, b) = 1 \Leftrightarrow \forall_{x \in D}((x \mid a, b) \Rightarrow x \mid 1)$ and so the negation of "a, b are coprime" would be $\exists_{x \in D}((x \mid a, b) \wedge x \nmid 1)$. Yet the negation of $(a, b)_v = D$ gives only that $(a, b)_v \neq D$, and $(a, b)_v \neq D$ does not imply that a, b have a nonunit common factor. For example in the ring $D = F[[X^2, X^3]]$ where F is a field, $(X^2, X^3)_v \neq D$ but $\text{GCD}(X^2, X^3) = 1$. On the other hand in some integral domains, such as GCD domains the notions of coprime and v-coprime coincide. I must note that v-coprimality is the ring theoretic equivalent of orthogonality (disjointness) in directed p.o. groups. Recall that a, b in a p.o. group $(G, \leq, \cdot)$ are disjoint if, $\inf(a, b) = a \wedge b$ exists and is e the identity, or equivalently $\sup(a, b) = a \vee b$ exists and is $a \cdot b$ [30, page 156] (We use orthogonal as a synonym of disjoint as we deal only with positive elements.) Indeed given $G(D) = \{kD{:}k \in K\backslash\{0\}\}$ partially ordered by $aD \leq bD$ if and only if $bD \subseteq aD$ for $a, b \in K\backslash\{0\}$, $G(D)$ represents the group of divisibility of D. Let $G(D)^+$ represent the positive cone of $G(D)$; then for $rD, sD \in G(D)^+$, $rD \vee sD \in G(D)$ translates to $rD \cap sD$ being principal, and if $rD \cap sD = rsD$ then in $G(D)$ we have $rD \vee sD = rDsD$ which forces rD and sD to be disjoint in $G(D)$.

Let $*$ be a general star operation and call $x, y \in D$ $*$-coprime if $(x, y)^* = D$. Since $A^* \subseteq A_v$ for every star operation $*$, we know that x, y being $*$-coprime implies x, y v-coprime but not conversely. For example in $F[X, Y]$ where F is a field, $(X, Y)_v = D$ but $(X, Y) = (X, Y)_d \neq D$. Finally no proper $*$-ideal contains a pair of $*$-coprime elements.

Proposition 1. *For a general star operation $*$ on $F(D)$ the following hold: (1) r, $s \in D$ are $*$-coprime to $x \in D$ if and only if $(rs, x)^* = D$. (2) For $r_1, r_2, ..., r_n \in D$, $(r_1 r_2 ... r_n, x)^* = D$ if and only if $(r_i, x)^* = D$. (3) $(r, x)^* = D$ if and only if every factor of r is $*$-coprime to x.*

Proof. (1). Suppose r and s are $*$-coprime to x and consider $(x, rs)^* = (x, rx, rs)^* = (x, (rx, rs)^*)^* = (x, r(x, s)^*)^* = (x, r)^* = D$. Conversely suppose $(rs, x)^* = D$ and consider $(x, r)^* = (x, rs, r)^* = ((x, rs)^*, r)^* = (D, r)^* = D$. Next, (2) and (3) follow from (1).

The following proposition lists some properties of $*$-coprime (and hence v-coprime) elements.

Proposition 2. *For a general star operation* $*$ *on* $F(D)$ *and for* $r, s \in D$ *the following hold. (i)* $(r,s)^* = D \Leftrightarrow (r^n, s)^* = D \Leftrightarrow (r^n, s^m)^* = D$ *for any natural* m, n*. (ii)* $(r,s)^* = D$ *and* $r|sy \Rightarrow r|y$*. (iii)* $(r,s)^* = D \Rightarrow D = D_r \cap D_s$*, here* $D_r = D_S$ *where* $S = \{r^n : n$ *ranges over nonnegative integers*$\}$*. (iv)* $(r,s)_v = D \Leftrightarrow D = D_r \cap D_s$*. (v)* $(r,s)_v = D$ *if and only if* r *and* s *do not share any prime* t*-ideals. (vi) Let* $x = \frac{r}{s} \in K \backslash D$*, if* s *has a nonunit factor that is* $*$*-coprime with* r *then* x *cannot be integral over* D*.*

Proof. (i) is direct. For (ii) note: $(r) = (r, sy)^* = (r, ry, sy)^* = ((r, y(r,s)^*)^* = (r,y)^*$. For (iii) let $(r,s)^* = D$ and consider $h \in D_r \cap D_s$. Then for some natural numbers m, n $hr^m, hs^n \in D$. So $D \supseteq (hr^m, hs^n)^* = h(r^m, s^n)^* = hD$ (by (i)). For (iv) use (iii) to establish that $(r,s)_v = D \Rightarrow D = D_r \cap D_s$. For the converse assume $D = D_r \cap D_s$ and note that $(r,s)^* = (r,s)D_r \cap (r,s)D_s$ $= D$. But $(r,s)^* = D \Rightarrow (r,s)_v = D$. For (v) note that if $(r,s)_v = D$ then r, s cannot be in a proper integral t-ideal and hence cannot be in a prime t-ideal. Conversely if r, s do not share a prime t-ideal then r, s do not share a maximal t-ideal. But then $(r,s)_w = \bigcap_{M \in t-Max(D)} (r,s)D_M = \bigcap_{M \in t-Max(D)} D_M = D$. But as $(r,s)_w \subseteq (r,s)_v$ we have the conclusion. For (vi) use the fact that if $\frac{r}{s}$ is integral over D then $s|r^n$ for some n; then use (i).

Using the aforementioned similarity between disjoint (or orthogonal) elements in (directed) partially ordered groups and v-coprime elements in integral domains we can associate with each nonempty $S \subseteq D \backslash \{0\}$ the m-complement $S^\perp = \{t \in D : (t,s)_v = D$ for all $s \in S\}$ and state the following proposition which comes from a recent paper by David Anderson and Chang [10].

Proposition 3. *Let* D *be an integral domain,* S, S_1 *and* S_2 *nonempty subsets of* $D \backslash \{0\}$ *and let* $\{S_\alpha\}$ *be a nonempty family of nonempty subsets of* D*. (1)* $S^\perp = (< S >)^\perp = (\overline{< S >})^\perp$*, where* $< S >$ *denotes the set multiplicatively generated by* S *and* $\overline{< S >}$ *denotes the saturation of* $< S >$ *. (We do not entertain empty sets nor empty products.) (2)* $S^\perp$ *is a saturated multiplicative set. (3) If* $S_1 \subseteq S_2$ *then* $(S_1)^\perp \supseteq (S_2)^\perp$*. (4)* $S \cap S^\perp \subseteq U(D)$ *where* $U(D)$ *denotes the set of units of* D*. (Equality if* $S \supseteq U(D)$*.) (5)* $S \subseteq (S^\perp)^\perp = S^{\perp\perp}$ *(notation). (6)* $S^\perp = (S^{\perp\perp})^\perp = S^{\perp\perp\perp}$ *(notation). (7)* $(\cup_\alpha S_\alpha)^\perp = (< \cup_\alpha S_\alpha >)^\perp = \cap (S_\alpha)^\perp$*. (8)* $(S_1 S_2)^\perp = (S_1)^\perp \cap (S_2)^\perp$*. (9) If* $S_1 \cap S_2 \neq \phi$ *then* $(S_1)^\perp (S_2)^\perp \subseteq (S_1 \cap S_2)^\perp$*. (10).* $D = D_{<S>} \cap D_{S^\perp}$*.*

The proofs are simple. For instance (1) and (2) can be proved using Proposition 1. Next, (3), (4) and (5) were treated in ([5, Proposition 2.4]). For (6), applying (3) to (5) we get $S^\perp \supseteq [(S^\perp)^\perp]^\perp$ and also applying (5) to $S^\perp$we get $S^\perp \subseteq [(S^\perp)]^\perp]^\perp$. In case of (7), for each α, $S_\alpha \subseteq \cup S_\alpha$ implies by (3) that $(S_\alpha)^\perp \supseteq (\cup S_\alpha)^\perp$ which in turn means that $\cap_\alpha (S_\alpha)^\perp \supseteq (\cup S_\alpha)^\perp$. For the reverse inclusion note that $x \in \cap_\alpha (S_\alpha)^\perp$ implies that x is v-coprime to each member of S_α for each α and so x is v-coprime to each member of $\cup S_\alpha$.

The equation in (8) can be established using Proposition 1 For (9) note that $\phi \neq S_1 \cap S_2 \subset S_i$ $(i = 1, 2)$ and so $(S_i)^{\perp} \subseteq (S_1 \cap S_2)^{\perp}$, but by (2), $(S_1 \cap S_2)^{\perp}$ is multiplicative and saturated.

As pointed out in [10] the inclusions in Proposition 3 can be proper. Our notation is different from that of [10]. This is partly in solidarity with [5] where it was noted that in the study of partially ordered groups $S^{\perp}$ is used to denote the set of elements orthogonal to elements of S, and partly because I want to see if some of the multiplicative ideal theory of domains can be used in studying partially ordered groups and monoids.

1.3 Applications of v-coprimality I (Divisibility)

The notion of v-coprimality is useful when we need to sift through factors in the presence of properties weaker than the GCD property. Recall that D is an almost GCD(AGCD) domain (monoid) if for each pair of nonzero elements x, y there is a natural number n such that $(x^n, y^n)_v$ is principal. The notion of AGCD domains was introduced in [41]. It was studied further in [8] and in [25]. It was shown in [8, Lemma 3.3] that D is an AGCD domain if and only if for each set $x_1, x_2, ..., x_n$ of nonzero elements of D there is a natural number m such that $(x_1^m, x_2^m, ..., x_n^m)_v$ is principal.

Here is a brief demonstration of the use of v-coprime elements. For this let us agree to call a quasi-local domain (D, M) t-local if the maximal ideal M is a t-ideal.

Proposition 4. *Let (D, M) be a t-local AGCD domain. Then for each pair $x, y \in D\backslash\{0\}$ there is a natural number n such that $x^n | y^n$ or $y^n | x^n$.*

Proof. Let $x, y \in D\backslash\{0\}$. If either of x, y is a unit we have nothing to prove. So, let $x, y \in M\backslash\{0\}$. Since D is AGCD, $(x^n, y^n)_v = dD$ for some natural n and $d \in D$. Or $(\frac{x^n}{d}, \frac{y^n}{d})_v = D$. Because M is a t-ideal, $\frac{x^n}{d}, \frac{y^n}{d}$ cannot both be in M, forcing one of $\frac{x^n}{d}, \frac{y^n}{d}$ to be a unit and making d an associate of x^n or of y^n.

The extent to which v-coprimality can be of use in bringing about uniqueness and order where there appears to be none is apparent in the study of factorization in integral domains that do not have the unique factorization property. Recall for instance that an integral domain in which every nonzero nonunit is expressible as a product of primary elements is called a weakly factorial domain (WFD). Now, every nonzero nonunit of a WFD can be written uniquely as a product of mutually v-coprime primary elements. Weakly factorial domains were introduced by Anderson and Mahaney [6] and further studied by Anderson and Zafrullah [7]. In a recent survey of “alternate” factorization in integral domains Anderson [1] treats WFD’s in greater detail. In a GCD domain, as mentioned earlier, the notions of coprime and v-coprime coincide, making the study of alternate factorizations much easier. It was in GCD domains that I started studying my kind of unique factorization. An element $r \in D$ is said to be a rigid element if r is a nonzero nonunit such that for all pairs $x, y \mid r$ we have $x \mid y$ or $y \mid x$. In [39] the following result was proved.

Proposition 5. *If in a GCD domain D, an element x is a product of finitely many rigid elements then x can be uniquely expressed as a product of finitely many coprime rigid elements.*

It is well known that D is a GCD domain if and only if its group of divisibility $G(D)$ is a lattice ordered group. Noting that the rigid element in a GCD domain is the same as the basic element ($b \in G^+, [0, b]$ is totally ordered) in a lattice ordered group. I decided to translate Conrad's condition F from [22] to the GCD domain setup as: A GCD domain D satisfies Conrad's condition F if every nonzero nonunit of D is divisible by at most a finite number of mutually coprime nonunits. Clearly, in a GCD domain, a nonzero nonunit that has no coprime factors is a rigid element. Using this I was able in [40] to prove the following result.

Proposition 6. *A GCD domain D is a ring of Krull type if and only if D satisfies Conrad's condition F.*

Recall from [32] that an integral domain D is a ring of Krull type if D has a family $\mathcal{F}$ of prime ideals such that for each $P \in \mathcal{F}$, D_P is a valuation domain and $D = \bigcap_{P \in \mathcal{F}} D_P$ is a locally finite intersection. The work on GCD domains satisfying Conrad's condition F came in handy when I became involved in a similar study of almost GCD domains in [25]. But in AGCD domains coprime and v-coprime do not coincide. For example a Dedekind domain with nonzero torsion class group is an AGCD domain but since such a domain is not a PID it must have a prime ideal P that is not principal and this forces P to have at least two non-associated irreducible elements x, y. Now being non-associated, x and y are coprime and being in P, x and y are not v-coprime, because being invertible P is a v-ideal. To cut the long story short we brought in new definitions. We called for a nonzero nonunit x the set $S(x) = \overline{< x >}$ the span of x and we called r an almost rigid element if for each m and for all $x, y \mid r^m$ there exists $n = n(x, y)$ such that $x^n \mid y^n$ or $y^n \mid x^n$. Thus in an AGCD domain an element r is almost rigid if and only if $S(x)$ contains no pair of v-coprime nonunits. We also showed that if r is almost rigid then the set $P(r) = \{x \in D{:}(r, x)_v \neq D\}$ is a maximal t-ideal. Calling 'of finite t-character' a domain in which every nonzero nonunit belongs to at most a finite number of maximal t-ideals we proved the following theorem.

Theorem 1. *An almost GCD domain D is of finite t-character if and only if for no nonzero nonunit $x \in D$, $S(x)$ contains an infinite set of nonzero nonunit mutually v-coprime elements.*

Indeed a ring of Krull type of Griffin [32] is a ring of finite t-character. In the AGCD (and hence GCD) situation an upper bound on the number of mutually v-coprime elements delivers some interesting results. The following result will facilitate the appreciation of those results.

Proposition 7. *If D is a domain with only a finite number of maximal t-ideals then D is a semi-quasi-local domain with each maximal ideal a t-ideal.*

Proof. Suppose that D has finitely many maximal t-ideals say $\mathcal{P}_1,\mathcal{P}_2,...,\mathcal{P}_r$; we can assume all of them to be distinct. Now since for every nonunit d in D, dD is contained in some maximal t-ideal we conclude that $\mathcal{P}_1\cup\mathcal{P}_2\cup...\cup\mathcal{P}_r$ consists of all nonunits of D. Next since every element of a nonzero prime ideal M is a nonunit we conclude that for each maximal ideal M we have $M\subseteq\mathcal{P}_1\cup\mathcal{P}_2\cup...\cup\mathcal{P}_r$. But then it is well known that M must be contained in one of the maximal t-ideals say $\mathcal{P}_i$ (see e.g. [33, Theorem 83]). But since M is maximal we have $M=\mathcal{P}_i$. From this it is easy to show that $\mathcal{P}_1,\mathcal{P}_2,...,\mathcal{P}_r$ are precisely the maximal ideals of D.

Recall that D is an almost Bezout domain if for each pair $a,b\in D\backslash\{0\}$ there is a natural number $n=n(a,b)$ such that the ideal (a^n,b^n) is principal.

Corollary 1. *An AGCD (a GCD) domain D having only a finite maximal set $S=\{x_1,x_2,...,x_n\}$ of mutually v-coprime (resp. coprime) nonunits is a semilocal almost Bezout (resp. Bezout) domain.*

Proof. The idea is that if there is a maximal set S of mutually v-coprime nonunits then these nonunits are each almost rigid. To each almost rigid element x_i we have a unique maximal t-ideal $P(x_i)$. Now these are the only maximal t-ideals of D. For if not and there is a maximal t-ideal P such that $P\neq P(x_i)$ for each i then $x_1x_2...x_n\notin P$. Then $(x_1x_2...x_n,P)_t=D$. That is there are $y_1,y_2,...,y_r\in P$ such that $(x_1x_2...x_n,y_1,y_2,...,y_r)_v=D$. This means that $x_1x_2...x_n,y_1,y_2,...,y_r$ do not share any prime t-ideals, which in turn means that $x_1x_2...x_n,y_1^m,y_2^m,...,y_r^m$ do not share any prime t-ideal where m is such that $(y_1^m,y_2^m,...,y_r^m)_v=pD$. But then $p\in P$ and so is a nonunit and we end up with $D=(x_1x_2...x_n,y_1^m,y_2^m,...,y_r^m)_v=(x_1x_2...x_n,(y_1^m,y_2^m,...,y_r^m)_v)_v=(x_1x_2...x_n,p)_v$. Consequently, $x_1,x_2,...,x_n,p$ are mutually v-coprime contradicting the maximality of S. Now we end up with finitely many maximal t-ideals and Proposition 7 applies. For the almost Bezout part let a,b be two nonzero elements of D; then since D is AGCD there is n such that $(a^n,b^n)_v=dD$. Or $(\frac{a^n}{d},\frac{b^n}{d})_v=D$. This means that $\frac{a^n}{d},\frac{b^n}{d}$ do not share any maximal t -ideal. But since all the maximal ideals are maximal t-ideals we conclude that $(\frac{a^n}{d},\frac{b^n}{d})=D$. But then $(a^n,b^n)=dD$.

Now here is an anecdotal proof of the fact that v-coprimality helps.

Proposition 8. *If an integral domain D contains two nonunits a,b such that $(a,b)_v=D$ but $(a,b)\subsetneq D$ then $Spec(D)$ is infinite.*

Proof. Exercise.

1.4 Applications of v-coprimality II (Splitting sets)

In this section we define and briefly describe the splitting and lcm-splitting sets and provide brief historical background on them.

Definition 2. *A saturated multiplicative set S of D is a splitting multiplicative set if each $x \in D\backslash\{0\}$ can be written as $x = ds$ where $s \in S$ and d is v-coprime to every member of S.*

It follows that if S is a splitting set then the m-complement $S^{\perp} = \{t{:}(t,s)_v = D,\ s \in S\}$ is also a splitting set. Note also that if S is a splitting set then $S \cap S^{\perp} = U(D)$. Here are a few characterizations of splitting sets whose proofs can be found in [2].

Theorem 2. *The following are equivalent for a saturated multiplicative set S: (1) S is a splitting set. (2) $< SD >$, the p.o. subgroup of $G(D)$ generated by $\{sD{:}s \in S\}$, is a cardinal summand of $G(D)$, the group of divisibility of D, i.e., there is a p.o. subgroup H of $G(D)$ such that $< SD > \oplus_c H = G(D)$. (3) If A is a principal integral ideal of D_S then $A \cap D$ is a principal ideal of D. (That is, principal integral ideals of D_S contract to principal ideals of D).(4) There is a multiplicative subset T of D such that (a) each element d of $D\backslash\{0\}$ can be written as $d = st$ where $s \in S$ and $t \in T$ and (b) any of the following equivalent conditions holds: (i) If $s_1t_1 = s_2t_2$, where $s_i \in S$ and $t_i \in T$ then $s_2 = s_1u$ and $t_2 = t_1u^{-1}$, where $u, u^{-1} \in D$. That is $d = st$ is unique up to associates. (ii) If $d = st$ $(s \in S, t \in T)$, then $dD_S \cap D = tD$. (iii) For each $s \in S$ and $t \in T$, $(s,t)_v = D$. (iv) For each $t \in T$, $tD_S \cap D = tD$.*

Some forms of the statements in Theorem 2 can be found in [31] and [34].

Definition 3. *A splitting set S is an lcm-splitting set if in addition every element of S has an lcm with every element of D.*

Proposition 9. *The following are equivalent for a saturated multiplicative set S: (i) S is lcm-splitting. (ii) $s_1D \cap s_2D$ is principal for $s_i \in S$. (iii) $s_1D \cap s_2D = sD$ for s, $s_i \in S$. (iv) $D_{S^{\perp}}$ is a GCD domain.*

Remark 1. It may be noted that if S in Proposition 9 is generated by primes then $D_{S^{\perp}}$ is a UFD [2, Proposition 2.6].

Since [2, Proposition 2.6] covers a lot of ground it seems best to quote it as a theorem.

Theorem 3. *The following conditions are equivalent for a saturated multiplicative set S of D: (1) S is generated by a set of prime elements $\{p_\alpha\}$ satisfying (a) for each α, $\bigcap_{n=1}^{\infty} p_\alpha^n D = 0$, and (b) for any sequence $\{p_{\alpha_n}\}$ of* *nonassociate members of* $\{p_\alpha\}$, $\bigcap_{n=1}^{\infty} p_{\alpha_n} D = 0$. *(2) S is generated by a splitting set of principal primes. (3) S is generated by a set of principal prime elements and S is a splitting set. (4) S is a splitting set and D_T is a UFD, where $T = S^\perp$ is the m-complement for S.*

(Note that a set $\{p_\alpha\}$ of principal prime elements is a splitting set of principal primes if the saturation of the set multiplicatively generated by $\{p_\alpha\}$ is a splitting set.) The set described in Theorem 3 is called a UF set in [34].

Using the statement of Definition 2 and using part (10) of Proposition 3 we can state the following result.

Proposition 10. *Let S be a splitting multiplicative set in D and let $T = S^\perp$ then $S = T^\perp = S^{\perp\perp}$ and $D = D_S \cap D_T$.*

Proposition 11. *Let S be a splitting set of D. (1) If P is a prime t-ideal, then P intersects S or P intersects $S^\perp$ but not both. (Any prime t-ideal that intersects both would be forced to contain a v-coprime pair, which is impossible.) (2) ([2]) If A is a nonzero ideal of D then $A_t D_S = (AD_S)_t$. (So, P is a prime t-ideal of D if and only if PD_S or PD_T is a prime t-ideal of the respective quotient ring.) (3) ([2]) Let $T = S^\perp$ and let $s_1, s_2, ..., s_m \in S$; $t_1, t_2, ..., t_m \in T$. Then $(s_1t_1, s_2t_2, ..., s_mt_m)_v = ((s_1, s_2, ..., s_m)(t_1, t_2, ..., t_m))_v$.*

The above proposition provides the sort of insight that gives you tools. For instance, from (1) we gather that if S is a splitting set it partitions the set of nonzero w-prime ideals into ones that intersect S and those that intersect $S^\perp$. Also look up [9]. From (2) we infer that if S is a splitting set then t-ideals of D extend to t-ideals of D_S, and v-ideals of finite type extend to v-ideals of finite type, something that usually does not hold. (For this see the discussion on pages 2522 and 2523 of [45].) Finally (3) is a gem. It shows that if S is a splitting set every v-ideal of finite type can be written as the v-image of a product of two ideals; one generated completely by elements from S and the other generated completely by elements from the m-complement T. Now assume that we are dealing with a t-invertible t-ideal A then we know that A is a v-ideal of finite type. So $A = (s_1t_1, s_2t_2, ..., s_mt_m)_v = ((s_1, s_2, ..., s_m)(t_1, t_2, ..., t_m))_v$ as a t-product of two t-invertible t-ideals. Now recall that under t-multiplication the group of t-invertible t-ideals modulo the group of nonzero principal ideals is called the t-class group of D, $Cl_t(D)$; if D is a GCD domain t-invertible t-ideals are principal and so $Cl_t(D) = 0$.These observations led the authors of [2] to prove the following result.

Theorem 4. *If S is a splitting set of D then $Cl_t(D) \simeq Cl_t(D_S) \times Cl_t(D_{S^\perp})$.*

In case S is an lcm-splitting set

$(s_1t_1, s_2t_2, ..., s_mt_m)_v = ((s_1, s_2, ..., s_m)(t_1, t_2, ..., t_m))_v$ becomes

$(s_1t_1, s_2t_2, ..., s_mt_m)_v = s(t_1, t_2, ..., t_m)_v$ where $s = (s_1, s_2, ..., s_m)_v$. Consequently we have the following result.

Theorem 5. *([2, Theorem 4.1]) Let D be an integral domain, S an lcm-splitting set, and T the m-complement for S. Then $D = D_S \cap D_T$, where D_T is a GCD domain. Every finite type integral t-ideal A of D has the form $A = s(AD_S \cap D) = s(t_1, t_2, ..., t_m)_v$ where $s \in S$, $t_1, t_2, ..., t_m \in T$. Moreover the map $Cl_t(D) \to Cl_t(D_S)$ given by $[A] \mapsto [AD_S]$ is an isomorphism.*

Splitting sets originated in efforts to produce generalizations of Nagata's theorem mentioned earlier. They first appeared in Gilmer and Parker [31] as Δ-sets, and in [34] as UF sets. Both the Δ and UF sets can now be described as splitting sets generated by prime elements. In [31] we also see [31, Proposition 2.2] stating some conditions that are equivalent to "x, y being LCM-prime (i.e. v-coprime)" and a statement ([31, Proposition 3.1]) which in the language of this and many recent papers can be rephrased as: If S is an lcm-splitting set of D and if D_S is a GCD domain then so is D. Then Mott and Schexnayder [34] gave the splitting sets the proper setting, they showed that a splitting set splits the group of divisibility of the domain into a cardinal product of two subgroups. Apparently a wish to give a more general form of Nagata's theorem had existed prior to [31], [34] and [38]. For example Samuel in [37] had restated Nagata's theorem for Krull domains and Cohn [20] had published his "Nagata's Theorem for Schreier domains". Indeed as pointed out in [31], Cohn had something like the following result in [21]: Suppose D is atomic (i.e. every nonzero nonunit of D is a finite product of irreducible elements) and S is a multiplicative set generated by some primes of D. If D_S is a UFD then so is D. Theorem 177 in [33] can also be cited as an example. In the next section I present, briefly, the current state of the art as far as applications and examples of splitting sets are concerned.

1.5 Splitting sets: Examples and Applications

In this section I plan to give examples of splitting sets along with their extreme cases and the various forms and generalizations of Nagata's theorem for UFD's.

Examples lend insight which is so very important for understanding. Understanding on the other hand causes further appetite for understanding, which would come from insight, and insight depends on examples and often reasoning. So, I bring in below some examples and some results that I hope will enhance the readers' understanding and at the same time whet the readers' appetite for more.

In a Noetherian (Krull, or atomic) domain D the saturation of every multiplicative set S generated by nonzero principal primes is an lcm-splitting set (look up [2, Corollary 2.7]). For general situations a good example of a splitting set is what Gilmer and Parker [31] (page 69) describe as a multiplicative set generated by a family of primes $\{p_\alpha\}$ in D such that no nonzero element of D is divisible by infinitely many members of $\{p_\alpha\}$ or by infinitely many powers of any member of $\{p_\alpha\}$. More generally in any domain the set $U(D)$ of units of D is a splitting set and so is $D\backslash\{0\}$. (Call these trivial splitting sets.) Here is a good example of how splitting sets can have an effect.

Theorem 6. *([7, Theorem]) An integral domain D is a weakly factorial domain if and only if every saturated multiplicative subset of D is a splitting set.*

Now here is an "appetite" question: All we have seen is an lcm-splitting set and trivial splitting sets. Is there a clear nontrivial example of a splitting set that is not an lcm-splitting set? For the answer, suppose that every nontrivial splitting set in the whole world (universe?) is an lcm-splitting set and let p be a prime in a Noetherian domain D. Also suppose that D is not a UFD. Now $S = \overline{<p>}$ is a splitting set and so is $S^\perp$ and by our assumption $S^\perp$ is lcm-splitting. But then by Theorem 3 D_S is a UFD. Since S is generated by a prime, by Nagata's theorem, we have that D is a UFD, a contradiction. On the basis of these arguments we can say the following.

Remark 2. There are sufficiently many splitting sets that are not lcm-splitting sets.

We have seen that there are plenty of splitting sets that are not lcm-splitting; however there are situations in which a splitting set has to be lcm-splitting. Of these one comes from [9] and it would look best stated as a theorem.

Theorem 7. *([9, Theorem 2.2]) Let S be a multiplicatively closed subset of D. If S is a splitting set of $D[X]$ then S is an lcm-splitting set in D and hence in $D[X]$. Conversely if S is an lcm-splitting set of D with m-complement T then S is an lcm-splitting set of $D[X]$ with m-complement $T^/ = \{f \in D[X]$: $(A_f D_T)_v = D_T\}$, where A_f denotes the ideal generated by the coefficients of f.*

Of immediate everyday interest is the following corollary:

Corollary 2. *D is a GCD domain if and only if $S = D\backslash\{0\}$ is a splitting set in $D[X]$.*

Ordinary splitting sets can come in handy in some other decision-making processes. For the next one we need to prepare a little. If $x \in D$ such that $x = a_1a_2...a_n$ where a_i are atoms, we say that x has an atomic factorization of length n. An integral domain D is a half factorial domain (HFD) if D is atomic and for each nonzero nonunit $x \in D$ the length $n = n(x)$ of atomic factorizations of x is fixed. The following results are to appear in [24].

Proposition 12. *Let $K \subseteq B$ be an extension of integral domains such that K is a field, $D = K + XB[X]$ and let $S = \{f \in D{:}f(0) \neq 0\}$. Then D is an HFD if and only if S is a splitting set.*

If, in the above proposition, the field K is replaced by a domain A that is not necessarily a field, we have a more interesting situation.

Proposition 13. *Let $A \subseteq B$ be an extension of integral domains, $D = A + XB[X]$ and let $S = \{f \in D : f(0) \neq 0\}$. Suppose that S is a splitting set of D and that each element of S has all factorizations of fixed length in D. Then D is an HFD.*

To appreciate something, we need to know about what it looks like and we need to know about what it does. We have already seen some of "what it does", and here we shall concentrate on Nagata-type theorems.

Theorem 8. *Let S be an lcm-splitting set in D generated by principal primes. Then D is a UFD (satisfies ACC on principal ideals, is atomic, is integrally closed, is completely integrally closed) if and only if the same holds for D_S.*

The proof(s) of the above Theorem schema can be picked from [3]. In [3] we essentially establish a link between the factorization properties of D_S with those of D when S is a splitting set generated by primes. The results on integral closure and complete integral closure came off the particular thought processes we had at that time.

Theorem 9. *Let S be a splitting set in D generated by principal primes. If D_S is a Mori domain, (Krull domain, PVMD, GCD domain, almost GCD domain), then so is D.*

The results stated for PVMD's and GCD domains and AGCD domains will follow from more general results (Theorem 10). For the Mori domains I cannot recall any references. So, I will just give a proof. First recall that D is a Mori domain if D has ACC on integral v-ideals. An integral domain D is a Krull domain if and only if D is completely integrally closed and Mori ([27]). (A UFD is well known to be a Krull domain.) It is known that a locally finite intersection of Mori domains is Mori [36]; for a quick reference see [42, Corollary 4]. Now let D_S be Mori, then by Remark 1, $D_{S^\perp}$ is a UFD which is Mori and $D = D_S \cap D_{S^\perp}$ an intersection of two Mori domains.

Theorem 10. *(Nagata type theorem for general lcm-splitting sets): If S is lcm-splitting, then D_S is a PVMD (GCD domain, AGCD domain) if and only if D is.*

For PVMD's, and GCD domains see [2, Theorem 4.3]. Now suppose that D_S is an AGCD domain. Then $D = D_S \cap D_{S^\perp}$ where D_S is AGCD and $D_{S^\perp}$ is GCD. Let a, b be two nonzero elements of D. Then $a, b \in D_S$ and so there exists n such that $((a^n, b^n)D_S)_v = hD_S$. Since S is lcm-splitting $hD_S = ((a^n, b^n)D_S)_v = (a^n, b^n)_v D_S$. But then by Theorem 5 $(a^n, b^n)_v = s((a^n, b^n)_v D_S \cap D)$ which is principal because S is a splitting set and $(a^n, b^n)_v D_S$ is principal. Conversely, it is easy to verify that if D is almost GCD and if S is a multiplicative set then D_S is AGCD.

Theorem 11. *(cf. [43, Corollary 1.5]) If D is a GCD domain and S a saturated multiplicative set of D then the $D + XD_S[X]$ construction is a GCD domain if and only if S is a splitting set of D .*

Let me note that [43, Corollary 1.5] has a problem. I do not mention that S must be a saturated set (as I do in Theorem 11). The oversight could have been caused by the fact that if S is a multiplicative set and if $\overline{S}$ the saturation of S then $D+XD_S[X] = D+XD_{\overline{S}}[X]$. Theorem 1 of [43] too can be restated as: Let D be a GCD domain and let S be a saturated multiplicative set in D. Then $D + XD_S[X]$ is a GCD domain if and only if for every PF-prime P of D with $P \cap S = \phi$ there exists $d \in P$ such that d is not divisible by any nonunit of S. (A PF prime in a GCD domain is just a prime t-ideal.) In [43, Corollary 1.5] too, replacing multiplicative S, saturated multiplicative S will do the trick. (For [43, Theorem 1], Evan Houston has suggested the following statement: Let D be a GCD domain and let S be a multiplicative set in D. Then $D + XD_S[X]$, is a GCD domain if and only if for every PF-prime P of D with $P \cap S = \phi$ there exists $d \in P$ such that d is not divisible by any nonunit of $\overline{S}$. I am thankful to Evan.) .

I end this section with an odd sort of result. The space constraints prevent me from making any statements about it, but the result is pretty interesting without any introductions.

Theorem 12. *([26]) Let S be an lcm-splitting set in a coherent domain D. If the integral closure of D_S is a GCD domain then so is the integral closure of D. Consequently, if D is Noetherian and if S is a splitting set generated by principal prime elements of D and if the integral closure of D_S is a UFD then so is the integral closure of D.*

Indeed it would be interesting to see if (a) this theorem can be put to some interesting use and (b) if some of the restrictions can be relaxed.

1.6 Generalizations of splitting sets

When a certain approach appears to be successful in one area, some researchers are tempted to see if it can be mimicked in another area or in the same area but in a different form. This is how notions get generalized. Generalization does not always have to be bad or trivial. Sometimes you generalize because you want to get a feel of what you are studying. Remaining in the box would only let you follow the beaten tracks, but a stroll outside the box has the potential of opening new doors. I am happy to announce that all the generalizations of v-coprimality and of splitting sets have proved to be useful in enhancing our understanding of divisibility.

The first generalization that I would like to present arose in connection with AGCD domains. The idea germinated in [25] but took real shape in [5], and has been further studied by Chang [17]. A saturated multiplicative set S of D is called an almost splitting set if for each $d \in D\backslash\{0\}$ we can find a natural number $n = n(d)$ such that $d^n = st$ where $s \in S$ and t is v-coprime to every element of S. Like the splitting sets almost splitting sets can be characterized as saturated multiplicative sets S such that for each $d \in D\backslash\{0\}$ $d^n D_S \cap D$ is principal for some natural number n. We can also define almost lcm-splitting sets as almost splitting sets S such that for $s \in S$ and for each (nonzero) $d \in D$ we have $s^n D \cap d^n D$ principal for some $n = n(s, d)$. It may be noted that in an AGCD domain an almost splitting set is automatically an almost lcm-splitting set. Like splitting sets the almost splitting sets are in abundance. For instance every saturated multiplicative subset of a Dedekind domain with torsion class group is an almost splitting set; see [5, Theorem 2.11] for a more general result. Here are a couple of results that can be stated within the setup of this survey.

Theorem 13. *([5, Theorem 3.12]) Let $S \subseteq D\backslash\{0\}$ be a saturated multiplicative set. Then $D + XD_S[X]$ is an AGCD domain if and only if (i) D and $D_S[X]$ are AGCD domains, and (ii) S is an almost splitting set.*

A quick corollary to this result is that if D is integrally closed AGCD domain and if S is a saturated multiplicative set, then $D + XD_S[X]$ is an AGCD domain if and only if S is an almost splitting set. This gives us a lot of examples of AGCD domains. Just to mention a general one, let D be a Dedekind domain with torsion class group and let S be a multiplicative subset of D. Then $D + XD_S[X]$ is an integrally closed AGCD domain that is also coherent. For the "coherent" part see [23, Theorem 4.32].

Theorem 14. *([17, Proposition 2.6]) Let D be integrally closed. Then $D\backslash\{0\}$ is an almost splitting set of $D[X]$ if and only if D is an AGCD domain .*

Of course there are points of difference; for instance if S is almost splitting $Cl_t(D)$ is no longer isomorphic to $Cl_t(D_S) \times Cl_t(D_{S^\perp})$ [17, Example 2.9].

The next generalization of splitting sets is an example of an essential generalization. As I mention in [45], I kept looking for a result that would allow me to construct a PVMD $D + XD_S[X]$ domain from D a PVMD. It did not happen until we hit upon the notion of a t-splitting set. A (saturated) multiplicative set S of D is a t-splitting set if for each nonzero nonunit $d \in D$ we have $(d) = (AB)_t$ where A and B are ideals with $A \cap S \neq \phi$ and B is such that $(B, s)_t = D$ for each $s \in S$. The t-splitting sets S are characterized by: If A is a principal ideal of D_S then $AD_S \cap D$ is t-invertible. In [4] we proved that for D a PVMD and S a multiplicative set of D the construction $D + XD_S[X]$ is a PVMD if and only if S is a t-splitting set of D. (GCD domains are a special case of PVMD's.) In [18] the t-splitting sets are further explored, and there we bring forth Nagata-type Theorems that do not seem to have anything to do with the GCD property or the UFD property. Here is a quick example:

Proposition 14. *[18, Corollary 3.8] Let X be an indeterminate over D, $G = \{f \in D[X]:(A_f)_v = D\}$ and let S be a nonempty multiplicative subset of G. Then $D[X]$ is a Krull (resp. Mori, integrally closed, completely integrally closed, essential, UMT, Prufer v-multiplication) domain if $D[X]_S$ is.*

I usually tend to think of the set $G = \{f \in D[X]:(A_f)_v = D\}$ as the Gilmer set, because I first saw Gilmer use it in [29] and since then I have often made good use of this set.

From t-splitting sets of elements we graduated to t-splitting sets of ideals in [19]. Let D be an integral domain, S a multiplicative set of ideals of D and $D_S = \{x \in K\colon xA \subseteq D$ for some $A \in S\}$ the S-transform of D in the sense of Arnold and Brewer[16]. If I is an ideal of D, then $I_S = \{x \in K\colon xA \subseteq I$ for some $A \in S\}$ is an ideal of D_S containing I. Denote by $S^\perp$ the set of ideals B of D with $(A+B)_t = D$ for all $A \in S$. Call $S^\perp$ the t-complement of S. Denote by $sp(S)$ the "saturation" of S (set of all ideals C of D such that $C_t \supseteq A$ for some $A \in S$). Call S a t-splitting set of ideals if every nonzero principal ideal dD can be written as $dD = (AB)_t$ where $A \in sp(S)$ and $B \in S^\perp$.

It turns out that S being t-splitting is equivalent to $sp(S)$ being t-splitting and that if S is generated by principal ideals and t-splitting then it is the usual t-splitting set defined above. Moreover if S is t-splitting then (i) so is $S^\perp$, and (ii) for each $C \in S$, C_t contains a t-invertible t-ideal of $sp(S)$. (So, a splitting set of ideals S is v-finite in Gabelli's terminology [28]) In fact if S_i is the set of all t-invertible t-ideals in $sp(S)$ then S_i is a t-splitting set with t-complement $S^\perp$.It turns out that a lot of results proved for (t-)splitting sets carry through to this more general setting albeit with some new interpretations. Here's a sampling of some of the results proved in [19].

Proposition 15. *Let S be a t-splitting set of ideals of D. Then for every nonzero ideal I of D we have $I_t = (AB)_t$ with $A \in sp(S)$ and $B \in S^\perp$, and this "splitting" of I is unique up to t-closures.*

Proposition 16. *Let S be a multiplicative set of ideals of D. Then S is t-splitting if and only if S is v-finite and every nonzero principal ideal of D_S contracts to a t-invertible t-ideal.*

Proposition 17. *A t-splitting set of ideals induces a natural cardinal product decomposition of the ordered monoid of fractional t-ideals of D under the t-product and ordered by the usual reverse inclusion.*

Finally here's something to remind you of the earlier "Nagata-type Theorems".

Proposition 18. *Let F be a family of height one t-invertible prime t-ideals of D such that every nonzero nonunit of D belongs to at most a finite number of members of F. Let S be a multiplicative set generated by members of F. Then the following hold: (i) D is a PVMD if and only if so is D_S. (ii) D is a Krull domain if and only if so is D_S. (iii) D is of finite t-character if and only if so is D_S.*

Now, a word about a gap that needs to be filled. In jumping from splitting sets to t-splitting sets we overlooked the possibility of studying, say, d-$*$-splitting sets, d for divisibility. It appears to me that there is a whole world of results parallel to those we know about splitting sets. Let me give an example. Call a saturated multiplicative set S a d-d-splitting set if every element $x \in D\backslash\{0\}$ can be written as $x = st$ where $s \in S$ and t is d-coprime to every member of S. Recall that d-coprime $\equiv$ comaximal. Example: A saturated multiplicative set S generated by height one principal maximal ideals such that no nonzero member of D is divisible by an infinite set of nonassociated primes from S.

Proposition 19. *If S is a d-d-splitting set generated by height one principal maximal ideals. Then D is a PID (Noetherian, Prufer) if and only if D_S is.*

The proofs are straightforward and so are left to the reader, but I hope the point is made.

Here are some papers that I could not include because given the space I could not do justice to them (different jargon) or I came to know about them so late in the day that finding a suitable section was too hard. For the use of splitting sets in the direct sum decomposition of the ideal class group of a Dedekind domain see [14]. For splitting sets in the locally half factorial (D_x is half factorial for each $x \in D\backslash\{0\}$) setup see [13]. The splitting sets also show up in the study of elasticity of factorization in [15]. Recently, in [11] the notion of homogeneous splitting sets has been introduced, in the graded domain environ. Finally, on seeing an earlier version of this paper, Chang has sent me a preprint of a recent paper of his with David Anderson and Jeanam Park [12] that contains a general theory of splitting sets. For a finite character star operation $*$ they call a saturated multiplicative set S a g^*-splitting set if each $d \in D\backslash\{0\}$ can be written as $d = st$ where $s \in S$ and t is $*$-coprime with every element of S. It turns out that every g^*-splitting set is a splitting set but not conversely. They also study the $*$-complement of a subset $\phi \neq S \subseteq D\backslash\{0\}$ and redo quite a few results on splitting sets. Interesting reading.

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